Mathematics

Unit 2

Trigonometry

Acknowledgements:

Editor: Allison Kitto
Writer: Allison Kitto, Rob Houghton-Rhodes
Layout: Lidia Kruger

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LESSON 1

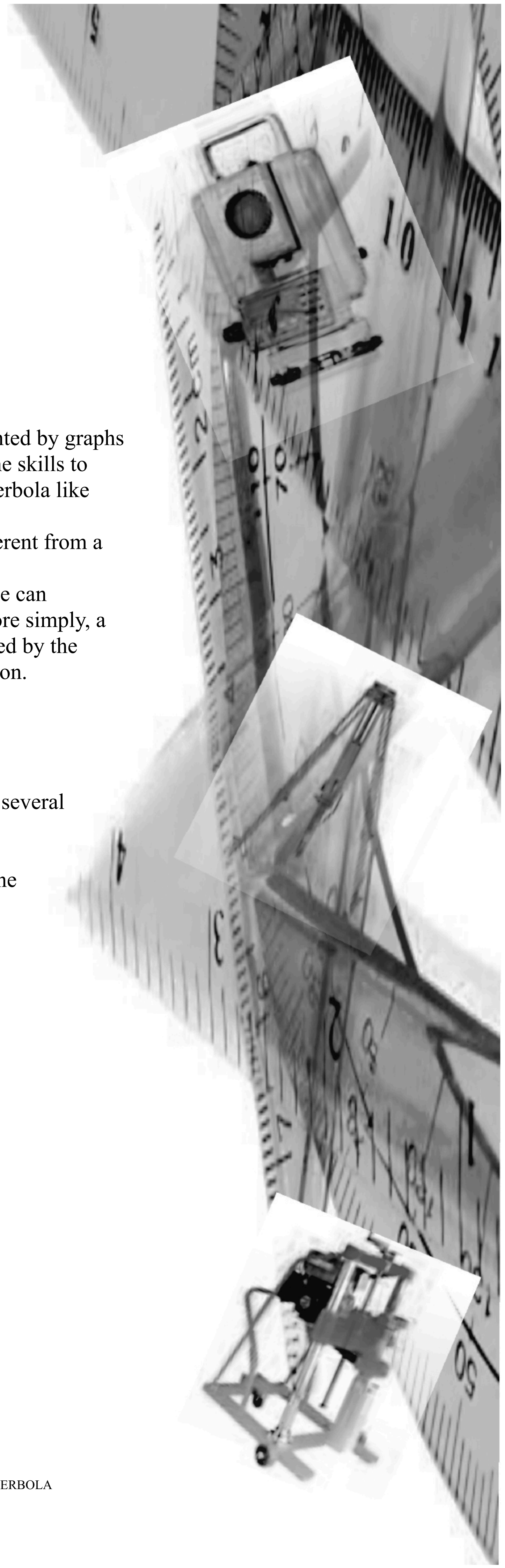
The Hyperbola

About this lesson

In Unit 1 we looked at relationships that can be represented by graphs and function notation. In this lesson we will use the same skills to learn about the hyperbola. If we start with a simple hyperbola like $y = f(x) = \frac{4}{x}$, we can see that the equation is very different from a straight line or a parabola. The equation of a parabola we can recognise by the presence of a quadratic function, or more simply, a squared unknown. The hyperbola's equation is recognised by the unknown (x) being part of the denominator of the function.

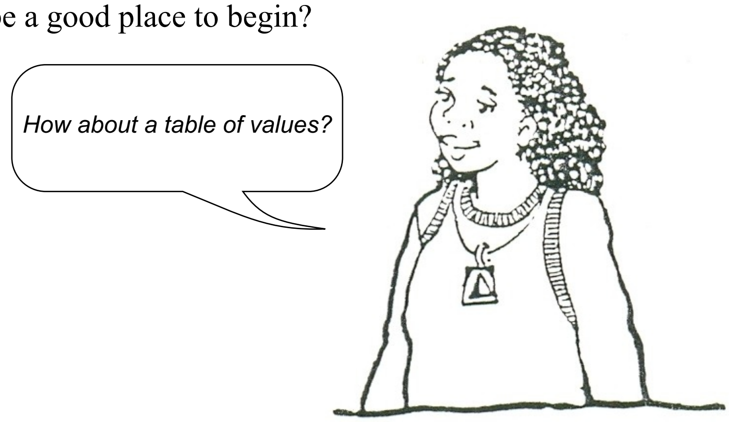
In this lesson you will:

- investigate the shape of the hyperbola
- learn to draw a hyperbola without needing to find several points along the curve
- discuss domain and range of the hyperbola
- transform the hyperbola within the coordinate plane



The Shape

If we want to know what this function looks like, where do you think would be a good place to begin?



Good, a table of values will always give us an idea of the shape of the graph.

Example

Populate the table below and use it to determine the shape of the graph

$y = f(x) = \frac{4}{x}$:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = \frac{4}{x}$									

Solution

To determine each y –value, we substitute the given x –value into the function.

When $x = -4$, the function f will have the value $y = \frac{4}{-4} = -1$.

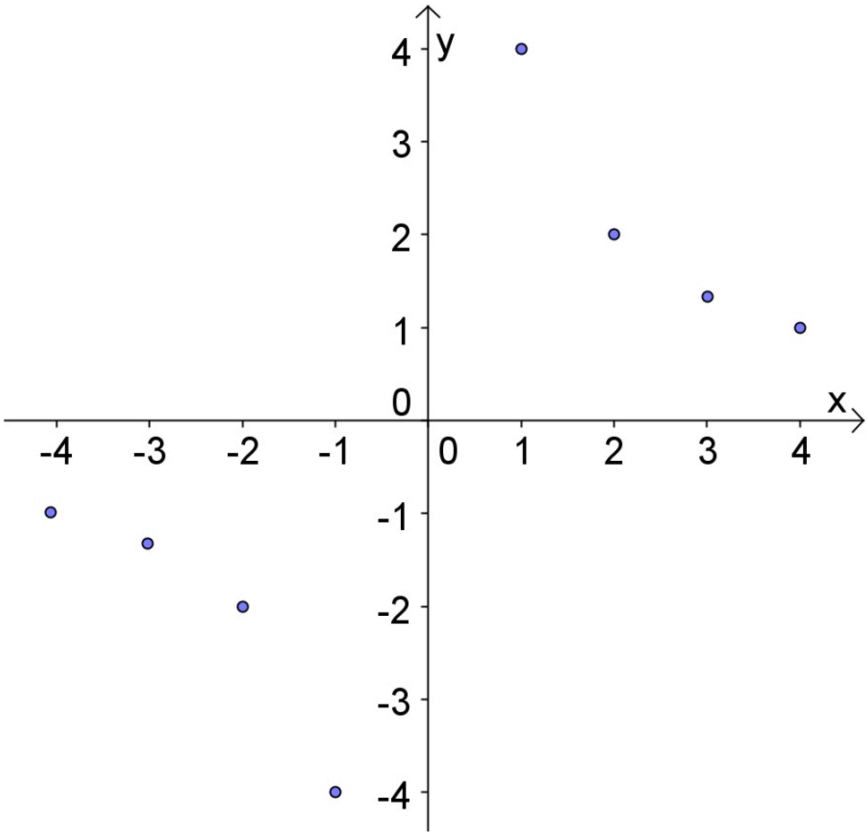
That is, $y = f(-4) = \frac{4}{-4} = -1$

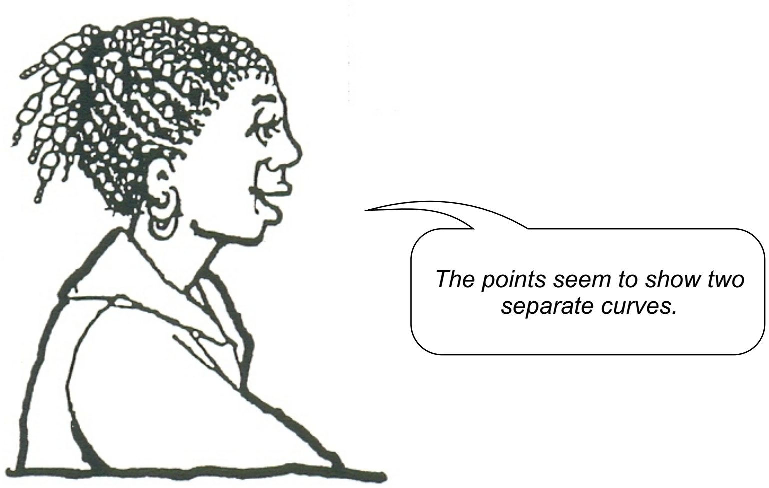
We do this for each of the x –values in order to populate the table.

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = \frac{4}{x}$	-1	$-\frac{4}{3}$	-2	-4	?	4	2	$\frac{4}{3}$	1

The problem we find when $x = 0$ is that we cannot divide by zero. The function is undefined when $x = 0$.

Let's sketch what we have and plan our next step.





Yes, that is correct. We need to have a closer look at what happens as the x –values get closer to zero. To do this, try the following activity.

ACTIVITY 1

Find the corresponding y –values for each of the following x –values where $y = \frac{4}{x}$. Round off your answers to two decimal places where necessary.

x	$-0,8$	$-0,6$	$-0,4$	$-0,2$	$0,2$	$0,4$	$0,6$	$0,8$
y								

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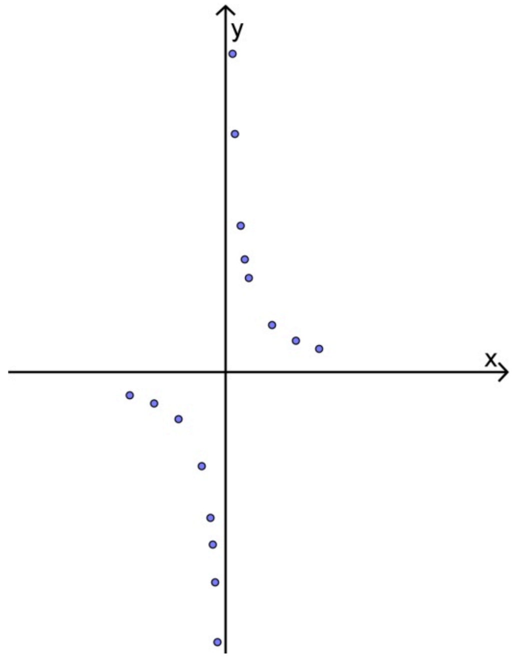
*An **asymptote** is a line or curve that shows a set of points which the graph gets closer and closer to, without touching or crossing it.*

If we add these points to the graph we created above, we get a clearer picture of the curve.

We can see that as the $\boldsymbol{x}$ –values get closer and closer to 0 from the right, the $\boldsymbol{y}$ –values get incredibly large and as we approach 0 from the left, the $\boldsymbol{y}$ –values get incredibly small. The points get closer and closer to the $\boldsymbol{y}$ –axis from both sides but can never touch it, as we know that the function is undefined when $x = 0$.

This line $x = 0$ is know as an **asymptote**.



Again the pattern indicates that as the $\boldsymbol{x}$ –values get very large or very small, the $\boldsymbol{y}$ –values get closer and closer to zero.

To check whether it will ever equal zero, we can substitute $y = 0$ into the function $y = \frac{4}{x}$

$\therefore$ *when* $y = 0$ *then* $\frac{4}{x} = 0$

$\therefore 4 = 0x$

$\therefore 4 = 0$

This is not true and so we know that there is no solution. There is no $\boldsymbol{x}$ –value that will make $(x = 0)$.

This means we have another asymptote. Both the $\boldsymbol{x}$ –axis $(y = 0)$ and the $\boldsymbol{y}$ –axis $(x = 0)$ are asymptotes. Our curve will approach these lines but never cross or touch them.

So the hyperbola has an asymptote where $x = 0$,the $\boldsymbol{y}$ –axis, which means it approaches this line but will never touch or cross it.

What happens when the $\boldsymbol{x}$ –values get very large and very small? Do the next activity to investigate this.

ACTIVITY 2

Find the corresponding $\boldsymbol{y}$ –values for each of the following $\boldsymbol{x}$ –values where $y = \frac{4}{x}$. Round off your answers to two decimal places where necessary.

x	-12	-10	-8	6	6	8	10	12
y								

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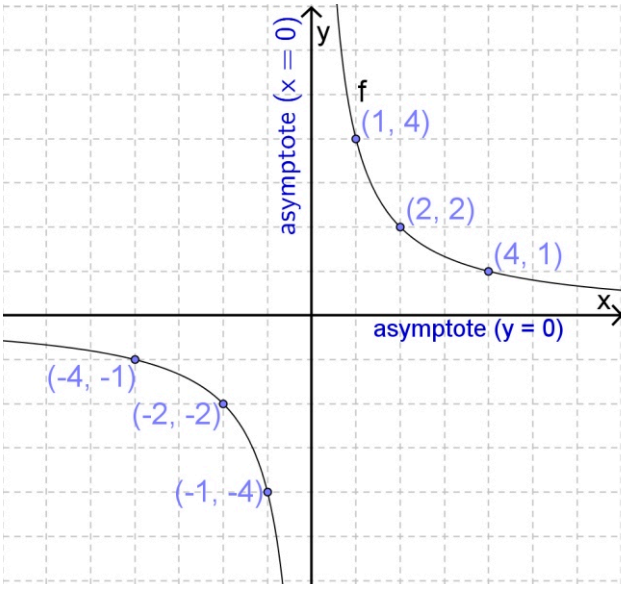
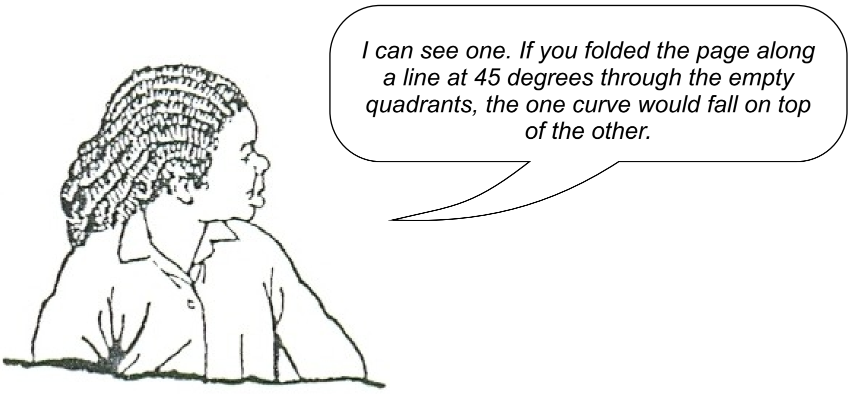
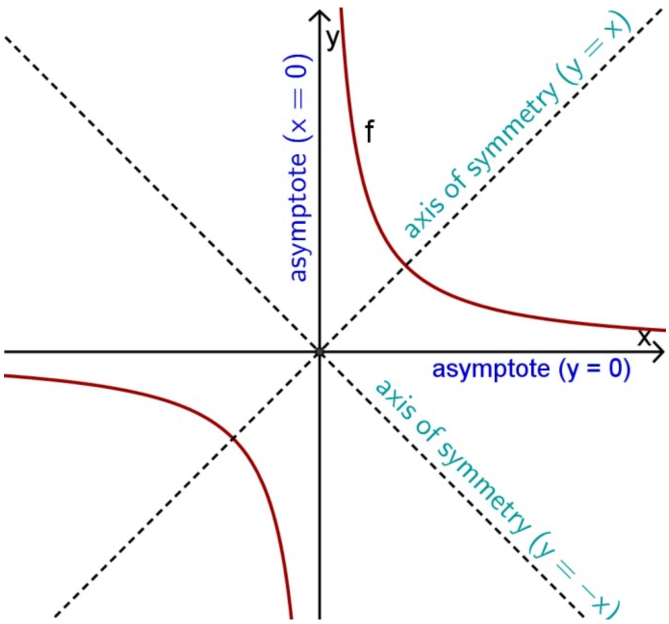
With all the points we have now and knowledge of the asymptotes, we can draw this graph.



If you look at the graph above, you will see that this graph, like the parabola, has an axis of symmetry. In fact, it has two. Can you see where the symmetry lies?



Good, that's right. The other axis runs through the same quadrants as the curve, also at 45°.



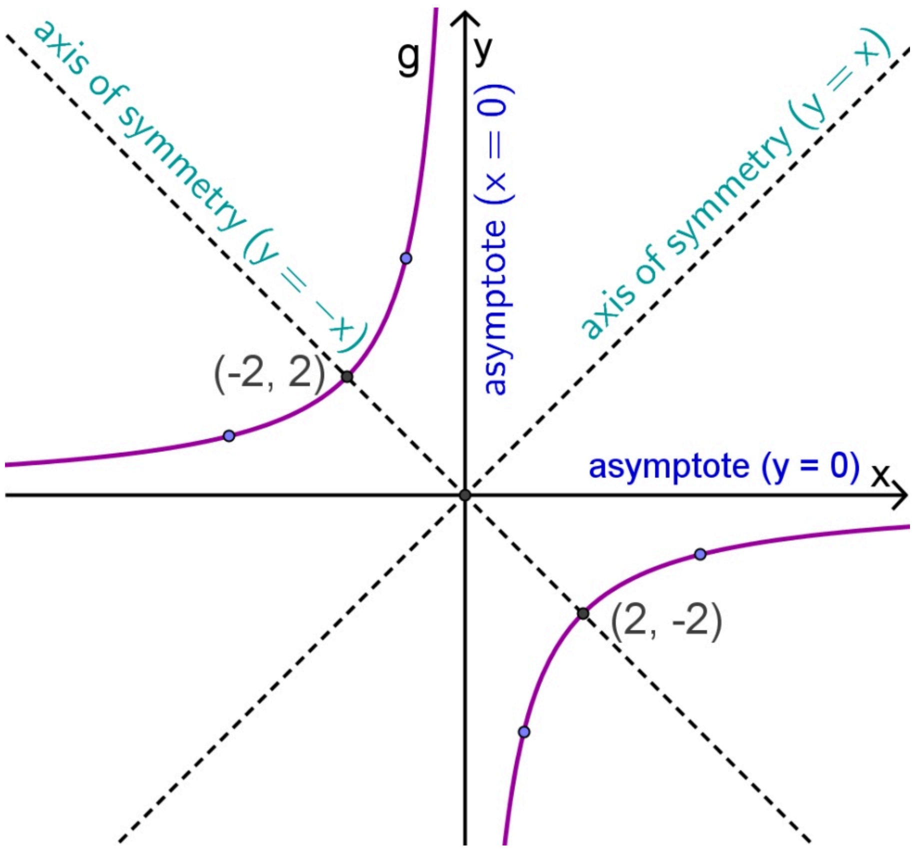
In order to generalise about a hyperbola, we need to look at some other equations.

Example
Using a table of values, draw a graph of the function $y = g(x) = -\frac{4}{x}$

Solution
We start with the table.

x	-4	-3	-2	-1	0	1	2	3	4
$y = g(x) = -\frac{4}{x}$	1	$\frac{4}{3}$	2	4	?	-4	-2	$-\frac{4}{3}$	-1

The values all look similar to the previous hyperbola. Remembering the two asymptotes and two axes of symmetry, we can draw the graph.



The general equation for a hyperbola is $y = \frac{a}{x}$. From the graph above, it would seem that when $a < 0$, the function is found in different quadrants to when $a > 0$.

The following activity will help us to make more generalisations.

ACTIVITY 3

Using a table of values, draw the following hyperbolas.

1. $y = \frac{1}{x}$
2. $y = -\frac{1}{x}$
3. $y = \frac{6}{x}$
4. $y = -\frac{2}{x}$

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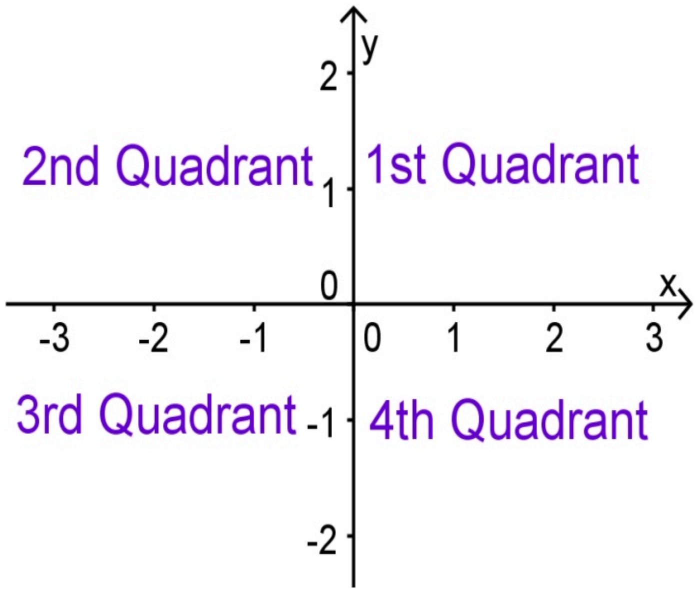
From your answers, we should be able to make a few deductions.

- Each hyperbola consists of 2 separate curves.
- In the general form $y = \frac{a}{x}$, if $a > 0$ then the curves are in the 1st and 3rd quadrant. If $a < 0$, then the curves are in the 2nd and 4th quadrants.
- The greater the value of ***a*** (ignoring the minus sign), the closer the curves will get to the origin.
- The product of each ***x*** and ***y*** coordinate pair that falls on the curve is equal to ***a***.



A table of values will always work to determine the shape of a function, however, if we can use our knowledge of the shape and find a few key points, we don't need to draw up the table.

Sketching Hyperbolas

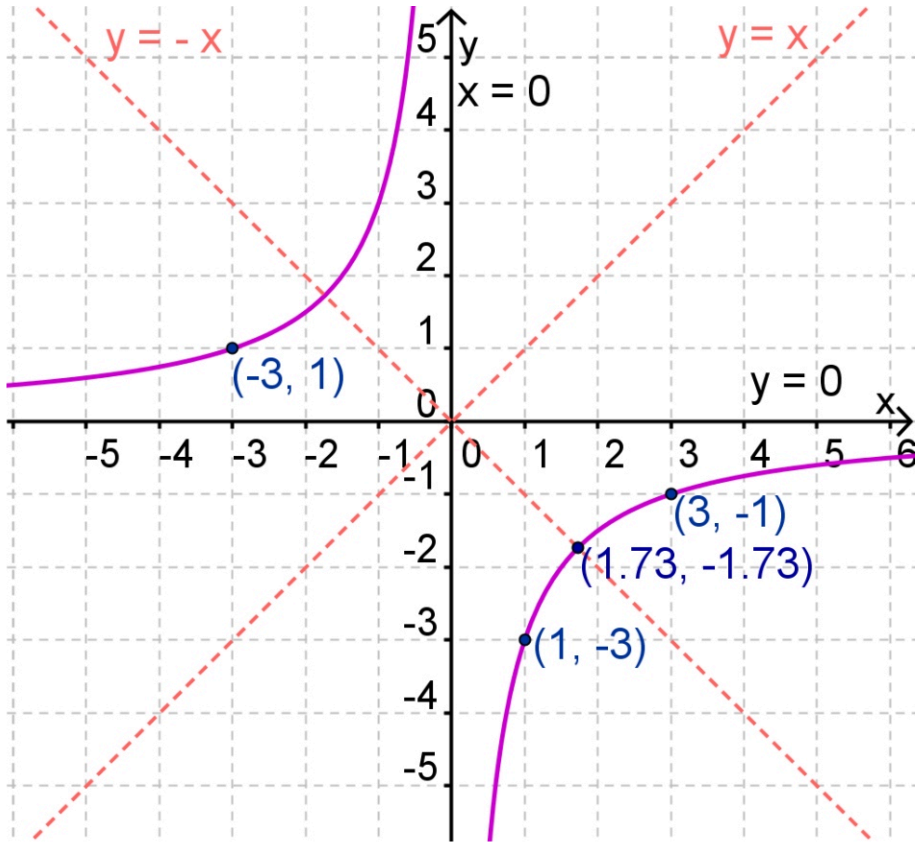
Using the deductions we have made from Activity 3, we can make some decisions about a hyperbola and sketch it from there. Let's try.

Example
Sketch the function $p(x) = -\frac{3}{x}$

Solution

1. We know that it's a hyperbola since the unknown (x) is part of the denominator.
2. $a = -3, \therefore a < 0 \therefore$ the curves will fall in the 2nd and 4th quadrants.
3. $a = -3 \therefore$ any point where the product of the $\boldsymbol{x}$ and $\boldsymbol{y}$ −coordinates is −3 will be on the curve, like $(-3;1), (3;-1), (1;-3)$ and $(\sqrt{3}; -\sqrt{3})$.
4. There are two asymptotes, $x = 0$ and $y = 0$.
5. There are two axes of symmetry, $y = x$ and $y = -x$.

Using these factors, we can sketch the graph.



Try some of these for yourself.

ACTIVITY 4

Sketch the following hyperbolas without first drawing up a table of values.

1. $y = \frac{9}{x}$
2. $y = -\frac{10}{x}$
3. $y = \frac{1}{2x}$ (hint: this is the same as $y = \frac{\frac{1}{2}}{x}$)

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Domain and Range

Consider the function $y = f(x) = -\frac{4}{x}$. This function has a set of

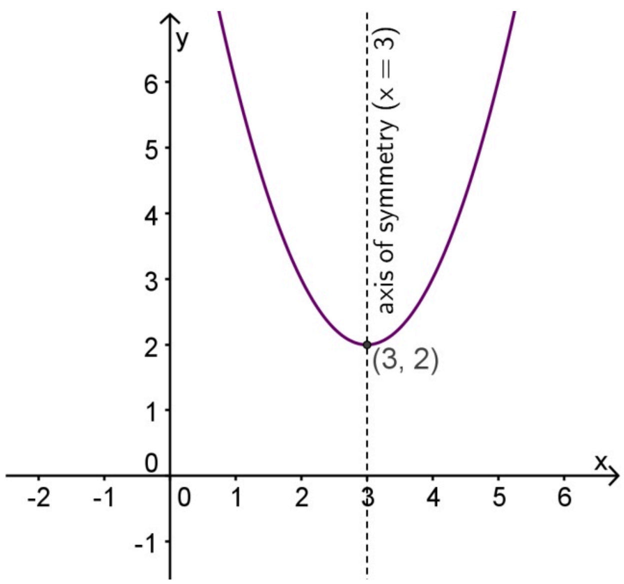
independent variables, in this case, a set of y –values. The set of x –values that result from substituting the set of x –values into the function, are the dependent variables, since their value depends on the value of x. The set of x –values is referred to as **domain** of the function and the set of y –values is said to be the **range** of the function.

Example
Determine the domain and range of the following function:
$y = (x-3)^2 + 2$

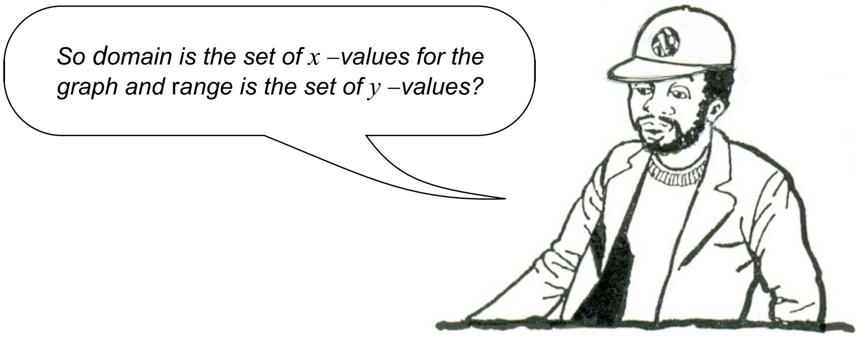
Solution
This function is a parabola. We can recognise it by its quadratic nature. Can you remember how to sketch this graph? If not, revise the lessons in Unit 1. For now, we are only interested in determining its domain and range, so I will sketch it for you.

From the equation we can see that the original parabola has shifted 3 units to the right and 2 units up and we can therefore find the turning point at $(3; 2)$.

The graph is symmetrical about the line $x = 3$
There are no x –intercepts
The y –intercept is at $y = 11$



When looking at domain and range, we have to ask ourselves 'for which values of x and y does this graph exist?'



Yes, that's correct. Looking at the graph, where, with respect to the x –values, does this graph start and stop?

Although we cannot see it on this sketch, we know that the shape of the parabola continues in both horizontal directions, both left and right and has no starting or ending point. The domain of this graph therefore, is $x \in \mathbb{R}$ or $x \in (-\infty;\infty)$.

The y –values, however, only begin at the turning point and then continue without limit. The range is therefore $y \in [2;\infty)$

Now let's try with a hyperbola.

Example

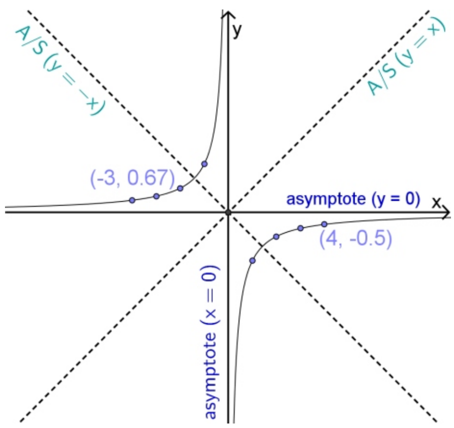
Determine the domain and range of the function $y = -\frac{2}{x}$

Solution

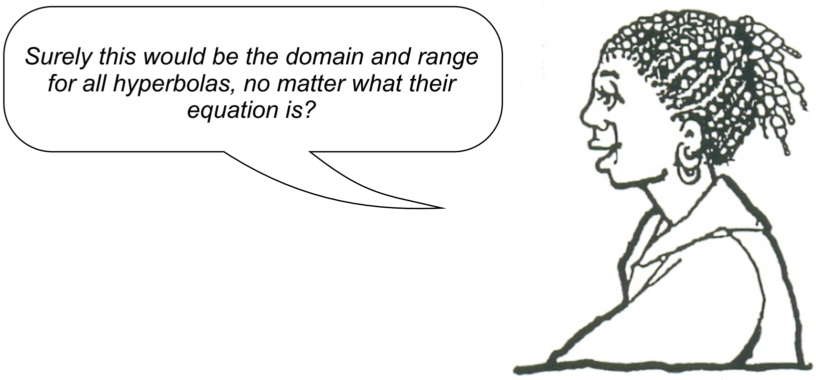
This is our sketch from earlier. What are the restrictions on the x – and y –values in this graph?

Although both parts of the function continue in both directions on the x –axis and y –axis, there are asymptotes which cannot be included.

Remember, the domain of a function is the set of all x –values for which the graph exists. We cannot, therefore, include $x = 0$, since the function is undefined for this value. The domain of this function is given by: $x \in \mathbb{R}$, $x \neq 0$



The same is true for the range. The graph does not exist when $y = 0$, so the range is given by: $y \in \mathbb{R}$, $y \neq 0$



As long as the hyperbola has not been translated, this is true.
But be careful, in the next section we are going to shift these graphs and then you are going to need to think clearly.

ACTIVITY 5

Determine the domain and range of the following functions. You will find it easier if you first sketch each function.

1. $y = -2x + 3$

2. $y = -(x-2)^2 - 1$

3. $y = \frac{3}{x}$

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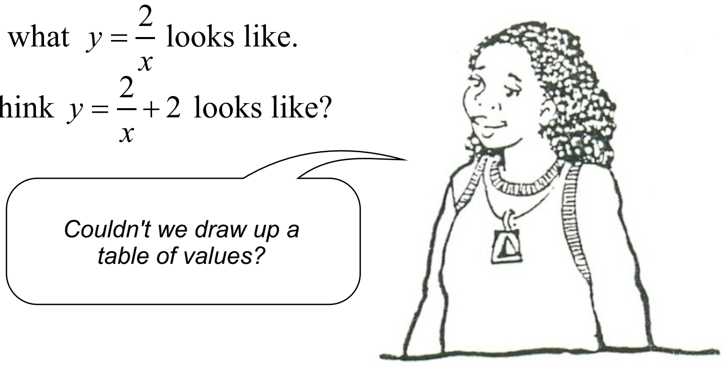
I think we're ready to start moving these hyperbolas around!

Transforming Hyperbolas

The hyperbola is shifted in very much the same way as the parabola.

We now know what $y = \frac{2}{x}$ looks like.

What do you think $y = \frac{2}{x} + 2$ looks like?



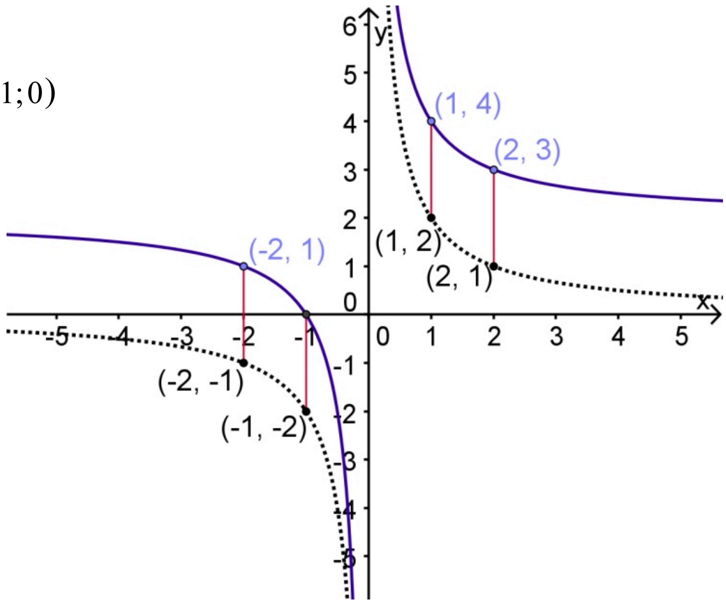
We could, yes, but since we have been through that process with the parabola, let's apply that knowledge to this hyperbola.

$y = \frac{2}{x}$ is a hyperbola in the 1st and 3rd quadrants. It goes through points like $(1;2)$, $(-1;-2)$ and $(\sqrt{2}; \sqrt{2})$. It has axes of symmetry at $y = x$ and $y = -x$ and asymptotes at $x = 0$ and $y = 0$.

By adding on +2, we are shifting this entire graph up by 2 units. All points, all symmetry and all asymptotes move with it. Can you picture it?

All points on the graph are translated up by 2 units.

$(1;2) \rightarrow (1;4)$
$(-1;-2) \rightarrow (-1;0)$
$(\sqrt{2};\ \sqrt{2}+2)$



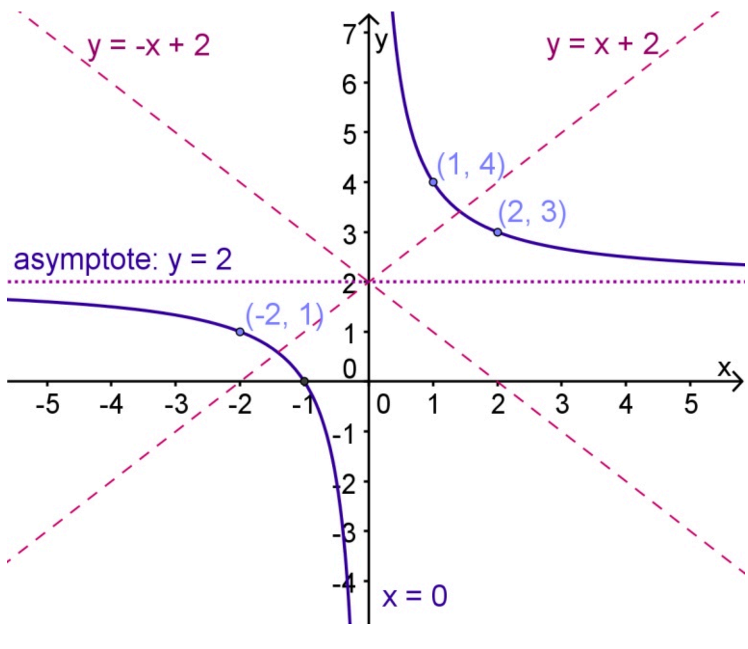
It may seem strange that we now have an x –intercept since the x –axis is supposed to be an asymptote.

But the asymptote has also moved up by 2 units as have both axes of symmetry.

Asymptote:
$y=0 \rightarrow y=2$

Axes of symmetry:
$y=x \rightarrow y=x+2$
$y=-x \rightarrow y=-x+2$

The vertical asymptote $(x=0)$ stays the same as the graph has not moved horizontally.



Once a hyperbola has been shifted vertically, it will have an x –intercept. It is always a good idea to calculate this intercept in order to place your graph accurately.

To find an x –intercept for the graph of any function, we let $y = 0$

$$y = \frac{2}{x} + 2$$
$$\therefore 0 = \frac{2}{x} + 2$$
$$\therefore -2 = \frac{2}{x}$$
$$\therefore -2x = 2$$
$$\therefore x = -1$$

The following activity uses graphs you have already drawn but now they include a vertical translation.

ACTIVITY 6

Calculate the x –intercept in each of the following functions and then sketch the function.

1. $y = \frac{9}{x} - 3$ 2. $y = -\frac{10}{x} + 5$

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The general equation for the hyperbola becomes $y = \frac{a}{x} + q$ where q represents the vertical shift.

As we can see with these graphs, the entire system is shifted up or down: the function, its asymptote and its axes of symmetry. The same is true with a horizontal shift.

Try to picture what the following function looks like. $y = f(x) = \frac{2}{x-1}$

This equation results in a horizontal shift. Again, the entire system shifts 1 unit to the right. This follows the same patterns as the parabola.

Yes, that is correct. With a horizontal shift, the asymptote has now moved away from the y –axis by 1 unit and is now sitting at $x = 1$. The axes of symmetry have also moved, as have all points along the curve.

With a horizontal shift, the horizontal asymptote $(y=0)$ does not move.

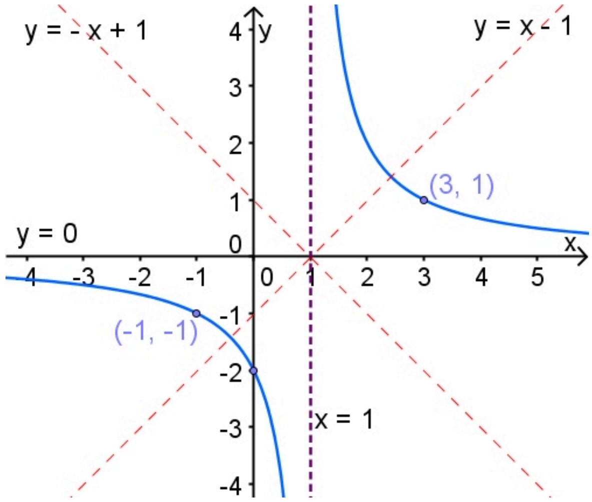
Let's calculate the y –intercept.

Let $x=0$ $\therefore y=\frac{2}{0-1}=-2$ $\therefore$ the y –intercept is -2.

Asymptotes:
$y=0$ stays the same
$x=0 \rightarrow x=1$

Axes of symmetry:
$y=x \rightarrow y=x-1$
$y=-x \rightarrow y=-(x-1)$
$\therefore y=-x+1$

Points on the curve:
$(2;1) \rightarrow (3;1)$
$(-2;-1) \rightarrow (-1;-1)$



Since the x –axis is still an asymptote, there is no x –intercept.

ACTIVITY 7

Calculate the y –intercept in each of the following functions and then sketch the function.

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1. $y=-\frac{4}{x+2}$ 2. $y=\frac{1}{x-2}$

The only thing to beware of with horizontal shifts is the direction. Unlike vertical translations, the direction of the shift is opposite to what we would expect.

With $y=-\frac{4}{x+2}$, the denominator reads $(x+2)$ which results in a translation to the left. The asymptote moves from $x=0$ to $x=-2$.

With $y=\frac{1}{x-2}$ the denominator is $(x-2)$ which results in a shift to the right. The asymptote moves from $x=0$ to $x=2$.

Here the general equation of the hyperbola is

$$y=\frac{a}{x-p}+q$$

where q represents the vertical shift and p represents the horizontal shift.

Dual Shifts

Of course it would be no fun at all if we didn't shift vertically and horizontally at the same time!

It's not as bad as it looks. Let's go through it.

Example

Sketch the function $y = \dfrac{4}{x+1} - 2$

Solution

The 4 tells us that this used to exist in the 1st and 3rd quadrants.
It has now been shifted 1 unit to the left and 2 units down.

Points on the curve have all shifted 1 unit left and 2 units down

$$(2;2) \rightarrow (2-1;2-2) = (1;0)$$
$$(4;1) \rightarrow (4-1;1-2) = (3;-1)$$
$$(-1;-4) \rightarrow (-1-1;-4-2) = (-2;-6)$$

Axes of symmetry have moved exactly how the graph moved.

$$y = x \quad \rightarrow \quad y = (x+1) - 2$$
$$\therefore y = x - 1$$

$$y = -x \quad \rightarrow y = -(x+1) - 2$$
$$\therefore y = -x - 3$$

Asymptotes are both affected.

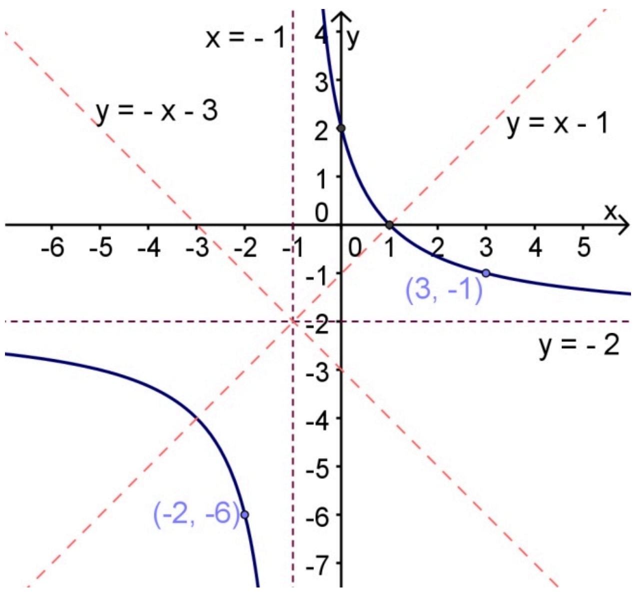
$$x = 0 \quad \rightarrow x = -1$$
$$y = 0 \quad \rightarrow y = -2$$

x –intercept: Let $y = 0$

$$\therefore 0 = \frac{4}{x+1} - 2$$
$$\therefore 2 = \frac{4}{x+1}$$
$$\therefore 2x + 2 = 4$$
$$\therefore 2x = 2$$
$$\therefore x = 1$$

y –intercept: Let $x = 0$

$$\therefore y = \frac{4}{0+1} - 2$$
$$\therefore y = 2$$



Now you can try.

ACTIVITY 8

Sketch the following functions, showing all intercepts with the axes, asymptotes, axes of symmetry and at least 2 points on the curve.

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1. $y = -\dfrac{1}{x-2} + 1$ 2. $y = \dfrac{2}{x+4} - 1$

Summary

You have looked at the shape and positioning of the hyperbola in detail. You can calculate where the asymptotes are and where the lines of symmetry are.
The general equation of a hyperbola is $y = \dfrac{a}{x-p} + q$ where q represents the vertical shift and p represents the horizontal shift.

The value of a determines in which two quadrant the hyperbola starts. When a hyperbola is translated, all points along the curve undergo the same transformation. Asymptotes and axes of symmetry are also transformed. Try the self-check exercise to make sure that you are following.

CHECKLIST

Are you able to:

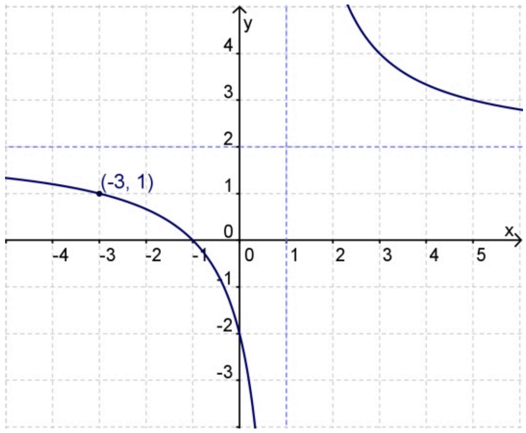
- investigate the shape of the hyperbola
- draw a hyperbola
- discuss domain and range of the hyperbola
- transform the hyperbola within the coordinate plane.

SELF-CHECK EXERCISE

1. Consider the function $y = \dfrac{3}{x-1} - 2$

   a) What are the x and y –intercepts of the above graph?
   b) Determine the equations of both asymptotes.
   c) Sketch the above function showing all intercepts with the axes, asymptotes, axes of symmetry and at least 2 points on the curve.
   d) What is the domain of the function?

2. Below is the graph of $y = \dfrac{a}{x-p} + q$. Determine the values of a, p and q.



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LESSON 2

Inequalities

About this lesson

This lesson is about inequalities – statements that show the relationship between two (or more) expressions with one of the following five signs: $<$, $\leq$, $>$, $\geq$, $\neq$. These are similar in form to the equations of equal quantities except we are dealing with unequal quantities.

The lesson introduces the concept of inequalities with variables, and shows how to find a solution set for an inequality, given a replacement set. It also introduces the formal properties of inequalities. It is necessary to know all the properties in this lesson in order to solve inequalities. In addition, these properties teach us more about the real numbers and how they interact with each other.

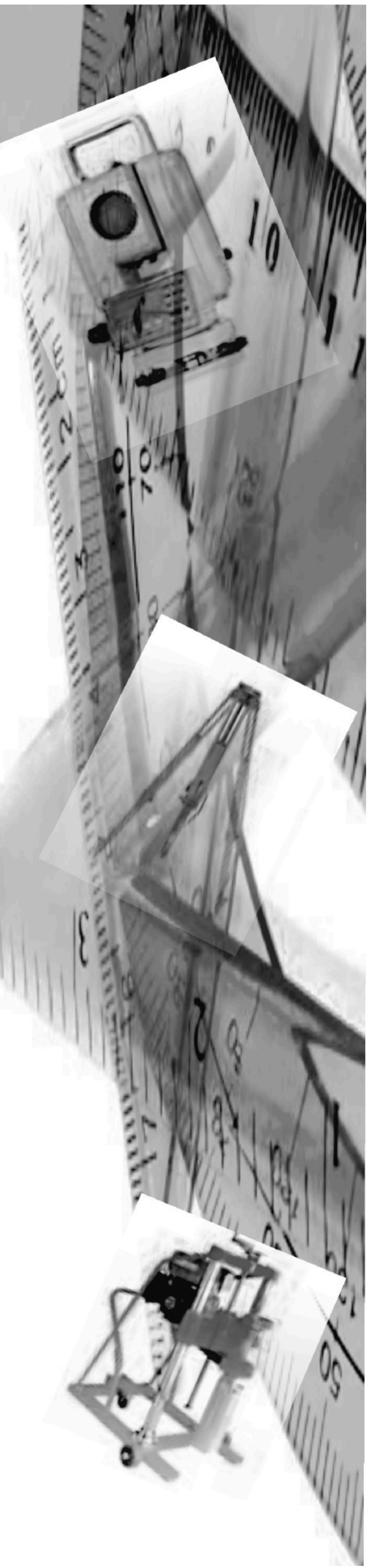
The lesson also shows you how to use inequalities to classify angles as right angles, acute angles, or obtuse angles.

Finally, any statement which includes the words 'at least', 'at most', 'more than', or 'less than' is an inequality. This lesson makes working with such statements easier, explaining what they mean in a mathematical sense, as well as how to figure out which numbers satisfy them and how to graph them.

Inequalities appear in a variety of fields – math, physics, chemistry, biology, economics, business – as well as in everyday tasks like cooking, spending money, and driving, for instance. Therefore, it is useful to understand them and to know how to work with them.

In this lesson you will:

- draw graphs of linear inequalities
- manipulate inequalities algebraically
- draw graphs of systems of linear inequalities in two variables
- solve quadratic inequalities

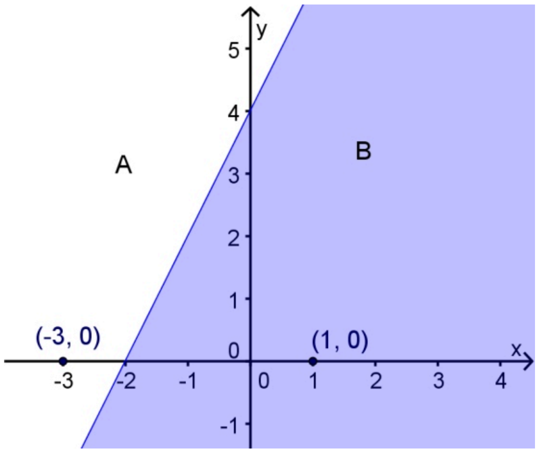


Linear inequalities

You have learned the symbols used in mathematics to show amounts that are:

smaller than (less than)	$<$	
bigger than (greater than)	$>$	
less than or equal to	$\leq$	
greater than or equal to	$\geq$	other amounts.

Take a look at this straight line graph.



Every point on the line graph obeys the rule $y = 2x + 4$
The straight line divides the plane into two regions, A and B.

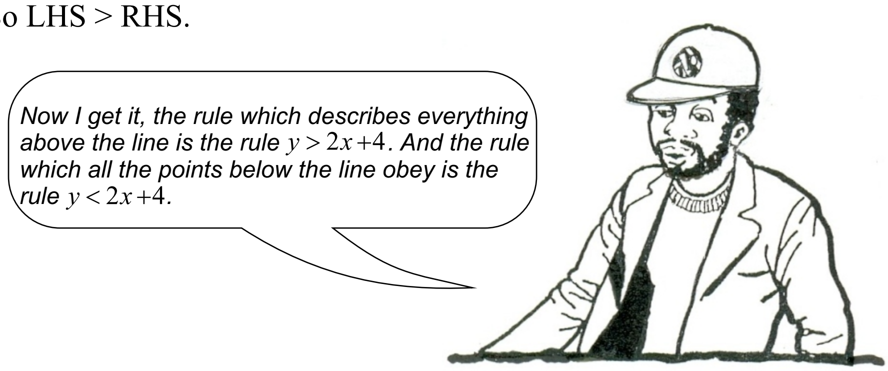
How would you describe the region A?

Well, for a start it includes all the points above the line.

Can you say what rule the points in A obey? Let us start by taking one point to see whether it obeys any rule. Let us take the point $(-3;0)$ and substitute this into the equation $y = 2x + 4$.

$$LHS = y \qquad\qquad RHS = 2(-3) + 4$$
$$= 0 \qquad\qquad = -2$$

As we can see, the left hand side is not equal to the right hand side.
Instead it is true to say $0 > -2$.
So LHS > RHS.



Yes, that is right. We could take a point in region B to check whether our rule is right. Let's take the point $(1;0)$ and substitute it into the equation $y = 2x + 4$.

$$LHS = y \qquad\qquad RHS = 2(1) + 4$$
$$= 0 \qquad\qquad\qquad = 6$$

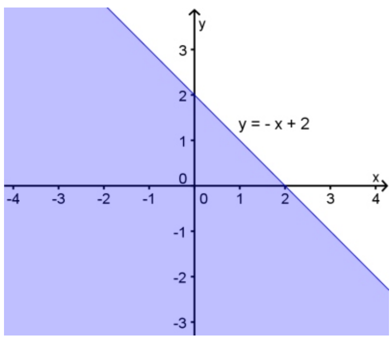
$\therefore$ LHS< RHS. So it is true that $y < 2x + 4$ for points in region B.

You will need to be able to define the shaded region in any sketch using the inequality notation.

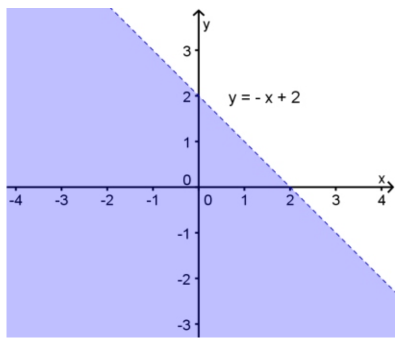
Example 1

Write down the rule which defines the shaded regions in each of the following sketches.

a)



b)



A broken line indicates that the points on the line itself are not included in the inequality.

Solution

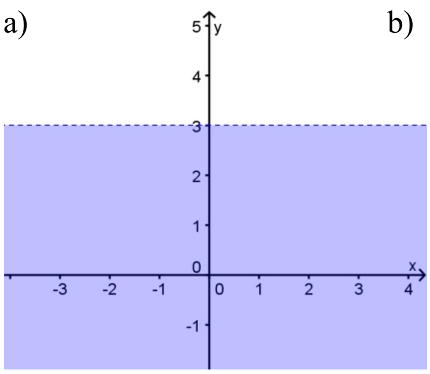
a) You are given the equation of the line to be $y = -x + 2$
To describe the shaded region we say $y \leq -x + 2$

b) You are given the same equation but the line is dotted this time so we must exclude the points on the line itself. Therefore the inequality is $y \geq -x + 2$

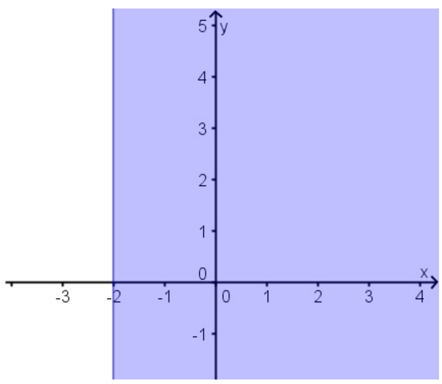
ACTIVITY 1

Write down the rules which define the shaded region in each of the following sketches.

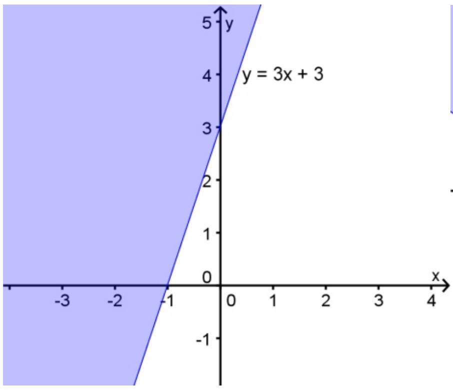
a)



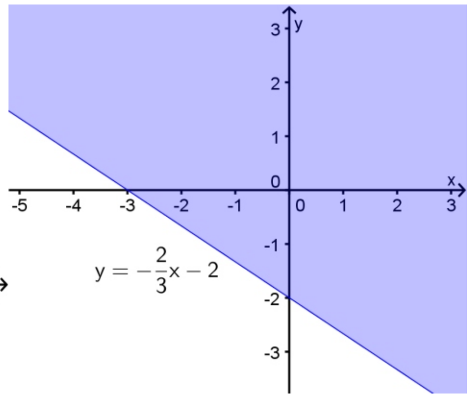
b)



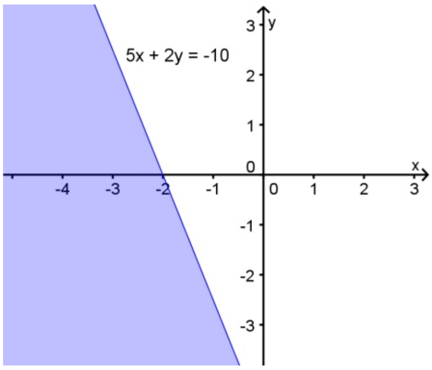
c)



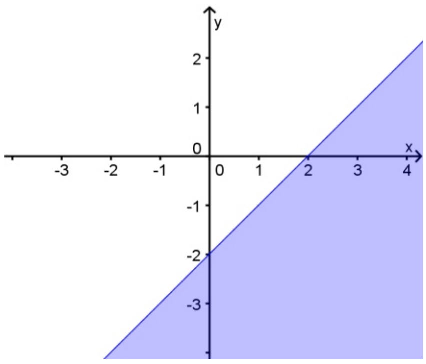
d)



e)



f)



ANSWERS ON PAGE 78

$y = mx + c$; all points on the line
$y > mx + c$; all points above the line
$y < mx + c$; all points below the line

We need to look at some properties of inequalities. After that we will look at methods of drawing inequalities such as $x - y > 3$.

Properties of inequalities

When working with inequalities you can use many of the algebraic techniques which you use for equations, but there are exceptions. It is wise to think carefully about what you are doing.

Example 2
Solve for x if $x - 2 > 4$

Solution
First we can add 2 to both sides. We get $x > 6$.

If we substitute a number greater than 6 back into the inequality we see that the answer is correct. There is not only one solution. There are many solutions to this inequality. In fact all numbers greater than 6 are part of the solution.

Let's try another example.

Example 3
Solve for $x + 3 < 6$

Solution
We subtract 3 from both sides. We get $x < 3$

In general, if you add or subtract equal quantities on both sides of the inequality, the inequality will still be true.

If we substitute any number less than 3 back into the inequality we see that the answer is correct.

Example 4
Solve for x in $2x > 6$

Solution
If we divide both sides by 2 we get $x > 3$

If we substitute any number greater than 3 back into the inequality, it will still be true.

Example 5

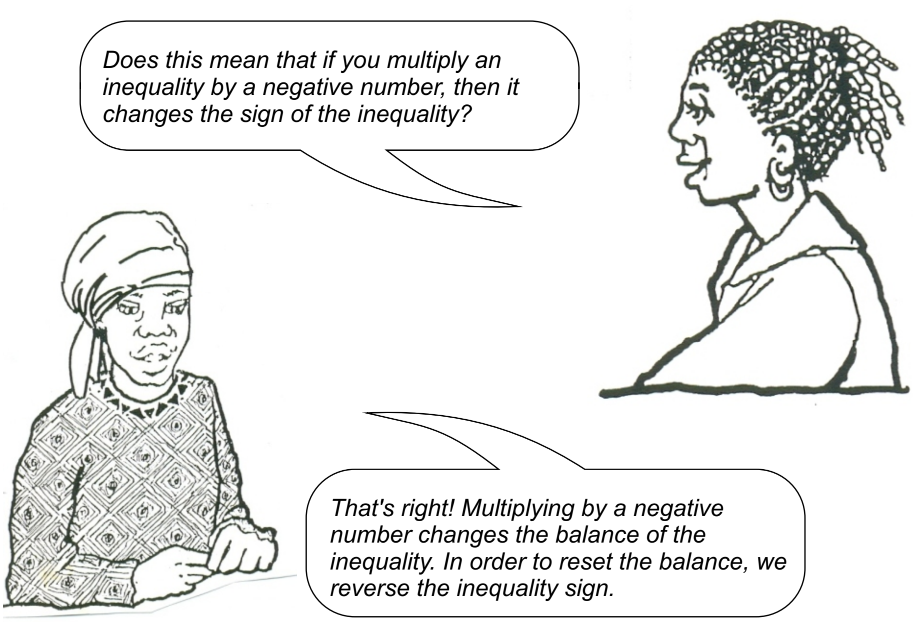
Solve for x if $-\frac{1}{2}x < 5$

Solution
$x < -10$ (multiply both sides by -2)

To check we substitute a number less than -10, say -12

$-\frac{1}{2}(-12) < 5$
$\therefore 6 < 5$

This is not true so we need to handle the negative differently. If we reverse the inequality sign, the statement will be true so our answer would then be $x > -10$.



Multiplying or dividing an inequality by a negative number changes the direction of the inequality sign.

Let's take a look at another example.

Example 6
Solve for x if $-2x > -10$

Solution
$x < 5$ (divide both sides by -2 and change direction of inequality sign)

Let's check, take $x = 3$ (which is less than 5) and substitute it into $-2x > -10$.

$-2(3) > -10$

$\therefore -6 > -10$ which is true.

So we can say that if you multiply or divide the inequality by a negative number, the inequality sign changes direction.

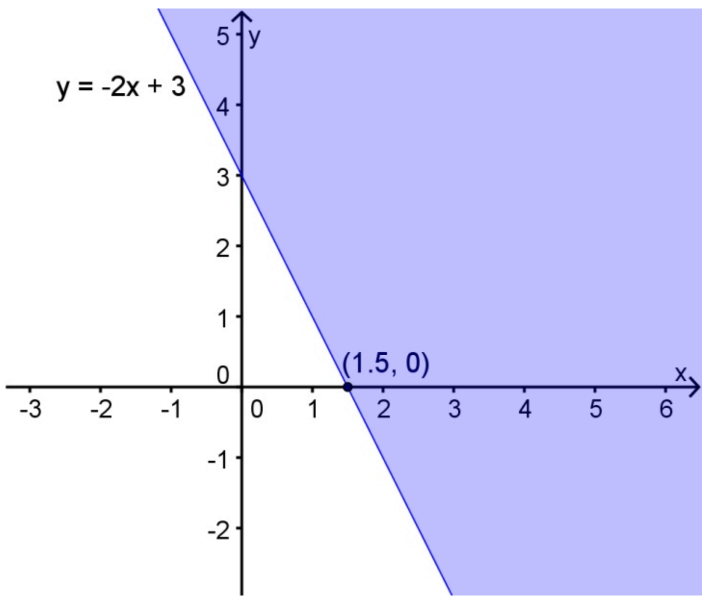
Drawing the graph of linear inequalities

You need to be able to draw straight lines with confidence in order to handle this section. If the inequality is given in the form $y > mx + c$ or $y < mx + c$ there is no real difficulty.

Example 7
Draw the graph $y \geq -2x + 3$

Solution
Draw the graph of $y = -2x + 3$ showing the intercepts on the axes, then shade the region $y \geq -2x + 3$

The solid line implies that all the points on the line are included in the inequality.



Example 8
Draw the graph of $4x - 2y > 1$

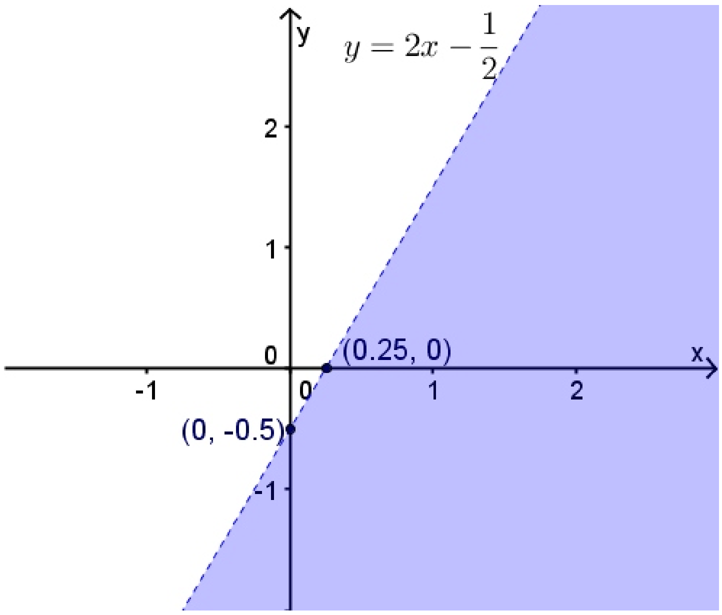
Solution
Change the inequality into standard form

$4x - 2y > 1$

$\therefore -2y > -4x + 1$ (subtract $4x$ from both sides)

$\therefore y < 2x - \frac{1}{2}$ (divide both sides by -2 and therefore change direction of inequality sign)

Draw the graph of $\therefore y = 2x - \frac{1}{2}$ as a dotted line, showing the intercepts on the axes, then shade the region $\therefore y < 2x - \frac{1}{2}$



The dotted line implies that that all the points on the line are ***not*** *included in the inequality.*

Remember the graph of an inequality is the set of all points $(x; y)$ which satisfy the inequality.

ACTIVITY 2

Draw rough sketches of the following inequalities

a) $y < -x + 2$
b) $y - x > 3$
c) $y \leq -2x$
d) $5x - 3y \geq 15$

ANSWERS ON PAGE 79

Systems of linear inequalities in two variables

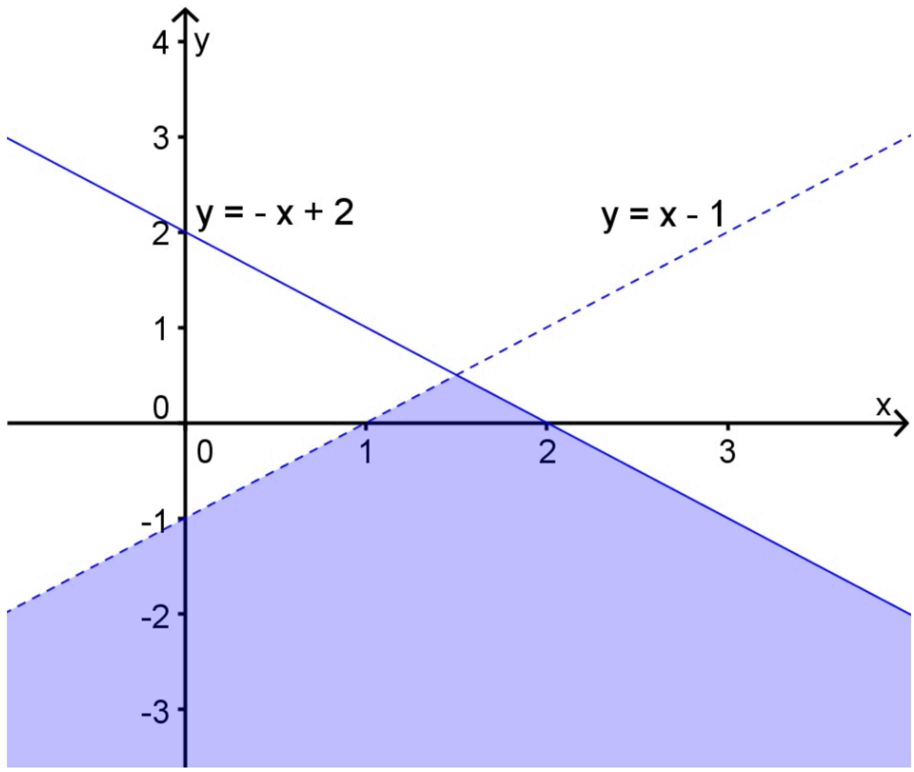
In the last few examples, you worked with single linear inequalities. We now want to take a look at two linear inequalities simultaneously.

Example 9

Graph the system of inequalities: $x + y \le 2$ and $x - y > 1$

Solution

$x + y \le 2$	(solid line)
$\therefore y \le -x + 2$	(shaded below)
$x - y > 1$	(broken line)
$\therefore -y > -x + 1$	(multiply both sides by –1 and reverse the direction of the inequality sign)
$\therefore y < x - 1$	(shaded below)



The region where the two inequalities overlap i.e. the area with the lines and the dots, is the set of points $(x; y)$ which satisfy both inequalities. Normally we shade only this area, which satisfies both inequalities so use a pencil when shading in the separate inequalities, then you can rub out the parts which do not satisfy both inequalities.

ACTIVITY 3

Graph these systems of inequalities.

Only shade the region which satisfies both inequalities.

a) $x + y > 3$
$x - y \geq 0$

b) $x + y \leq 4$
$y > 2$

c) $2x - y \leq 1$
$3x + y < 6$

d) $3x + 4y > 12$
$2x - 3y \leq 6$

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Solving Quadratic Inequalities

We've seen how to solve linear inequalities like $x + 7 > 14$. Now we need to consider quadratic inequalities like $x^2 - 16 > 0$. This is a more complex problem since the combination of factors decides the sign of the expression.

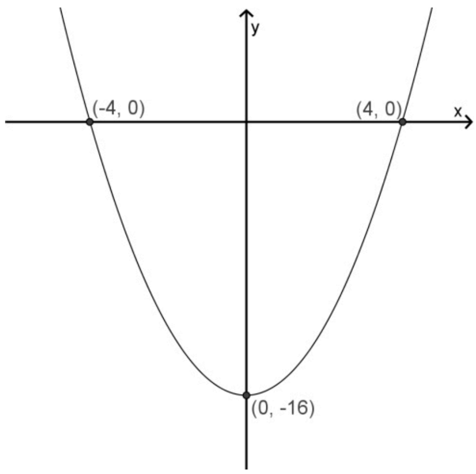
If we start in the same way as would to solve a quadratic equation, we would factorise the expression on the left hand side.

$x^2 - 16 > 0$

$\therefore (x-4)(x+4) > 0$

If both brackets are positive or both brackets are negative, then the expression will be positive. But if the brackets have different signs, then the expression will be negative. To move on from here we need to consider the parabola.

If we drew a graph of $y = x^2 - 16$, it would look like the sketch alongside.



This helps us to see the positive and negative nature of the expression. When the graph falls below the x –axis, the expression is negative and when it is above the x –axis, it is positive.

So to solve for x in $x^2 - 16 > 0$, we need only to decide when the graph of the expression would be above the x –axis.

$\therefore x < -4 \quad or \quad x > 4$

Where the graph cuts the y –axis is of no use to us in this solution, we need only find the intercepts with the x –axis in order to solve the inequality.

Let's look at another inequality.

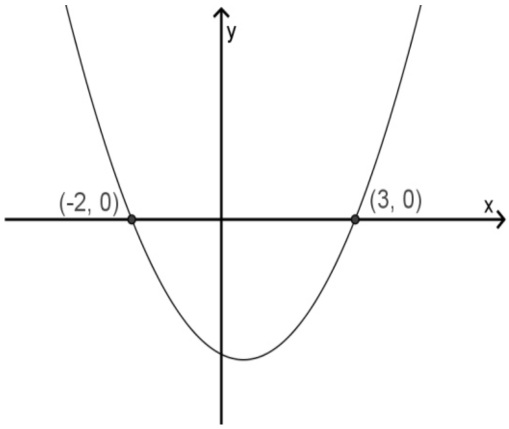
Example 10
Solve for x in $x^2 - 6 \leq x$

Solution
First we need all the terms on one side so that we can factorise the expression on the left.

$x^2 - x - 6 \leq 0$

$\therefore (x-3)(x+2) \leq 0$

Now we sketch the parabola.



The question asks for the values of x for which the expression is less than or equal to zero. From the sketch we can see that this will occur between –2 and 3 since this is when the graph falls below the x –axis. In this example we will include the two intercepts because this is where the expression is equal to zero.

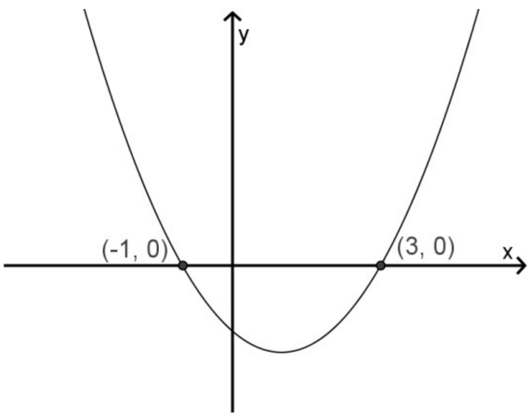
$\therefore -2 \leq x \leq 3$

Example 11
Solve for x in $-x^2 + 2x < -3$

Solution
$-x^2 + 2x + 3 < 0$
$\therefore x^2 - 2x - 3 > 0$ (multiply by a negative,
$\therefore$ reverse the inequality sign)

$\therefore (x-3)(x+1) > 0$

Now sketch the parabola.
$x < -1$ *or* $x > 3$
(where the graph is above the x –axis)



Try the activity below.

ACTIVITY 4

Solve for in each of the following:

1. $-2x-8 \geq -x^2$

2. $2x^2-3 < -5x$

3. $2(x^2+4)+10x \leq 0$

4. $7x-x^2 < 12$

ANSWERS ON PAGE 81

Summary: Drawing inequality graphs

1. When drawing inequality graphs, follow the same steps as drawing straight lines of the form $y=mx+c$.
2. First draw the graph by replacing the inequality sign with an equal sign.
3. If the inequality sign is > or < , the line of the graph is not included, and you draw a dotted line. If the inequality is ≥ or ≤ , the line of the graph is included and you draw a solid line.
4. For > or ≥ , shade above the line.
5. For < or ≤ , shade below the line.
6. Check using a point.

CHECKLIST

Are you able to:

- draw graphs of linear and quadratic inequalities on a Cartesian plane
- solve linear inequalities algebraically
- solve simultaneous linear inequalities graphically
- solve quadratic inequalities

SELF-CHECK EXERCISE

1. a) Draw the graph of $y=x+2$. Next, shade your diagram to show the $y \geq x+2$.
   b) Draw the graph of $y=-2x+3$. Shade in your diagram to show $y \leq -2x+3$.

2. Draw sketch graphs of the following:
   a) $y \geq 1$
   b) $y < x$
   c) $x \geq -3$
   d) $x < 2$

3. Draw the following system of inequalities:

$y > -x + 5$

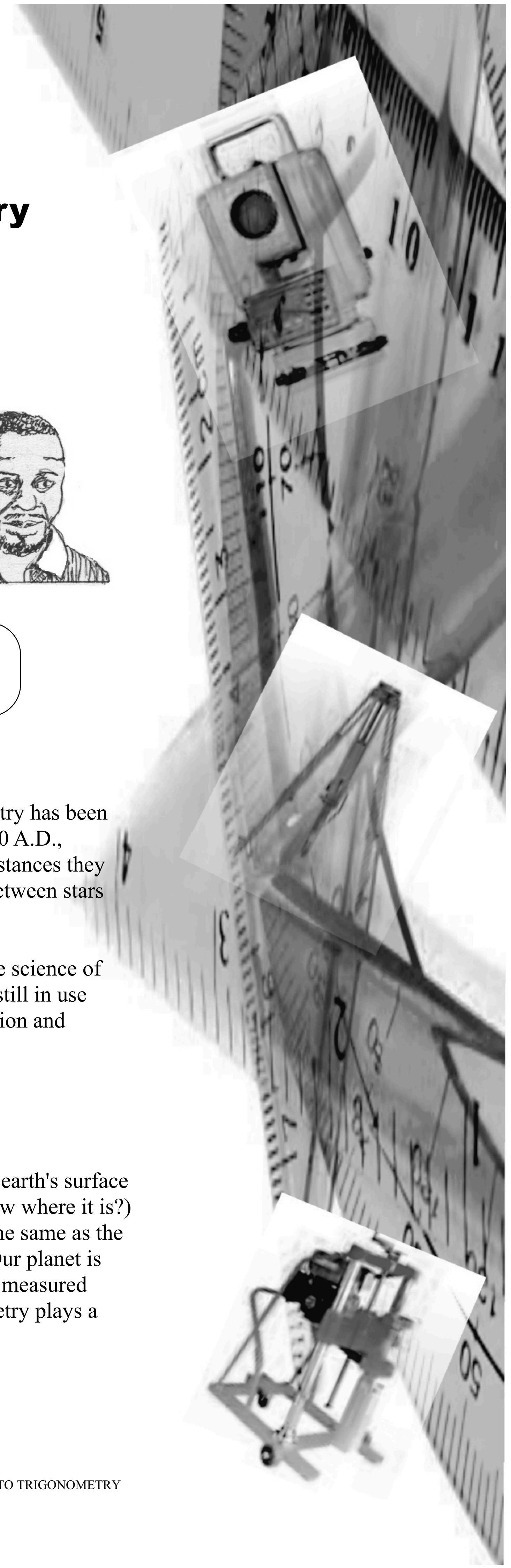
$2x - y < 4$

ANSWERS ON PAGE 94

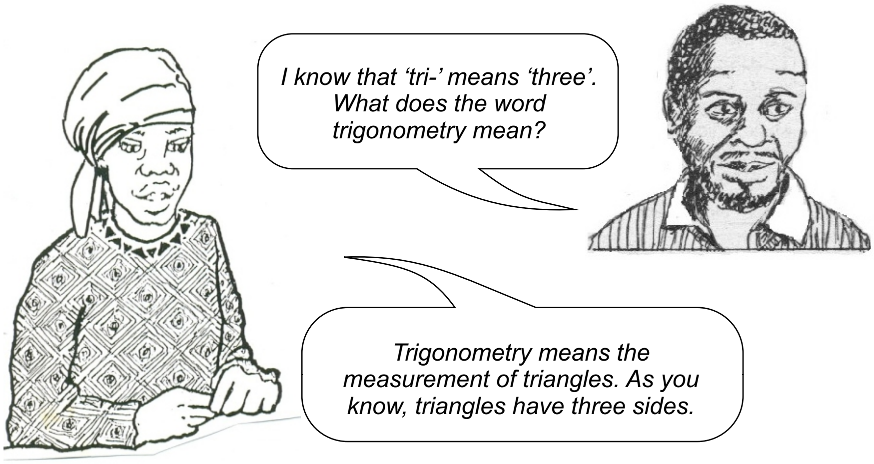
4. Solve for x in $-x^2 - 3x + 4 \geq 0$

LESSON 3

Introduction to trigonometry



About this lesson



Although the word is connected to triangles, trigonometry has been used far more widely than that. From about the year 150 A.D., Egyptians and Greeks used trigonometry to measure distances they could not measure directly. They measured distances between stars and the circumference of the earth.

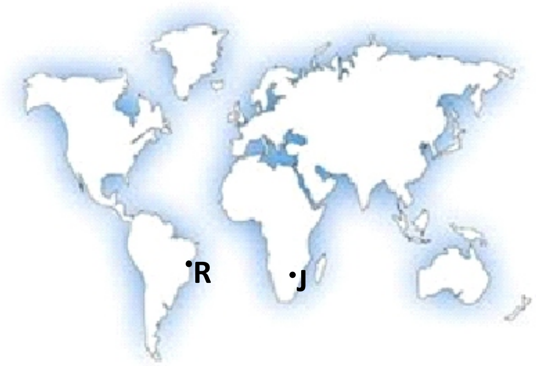
Remember, this information was used in navigation, the science of directing ships (and aircraft today). This knowledge is still in use today. We use solid trigonometry in astronomy, navigation and measurement of heights.

Let's take a look at an example!

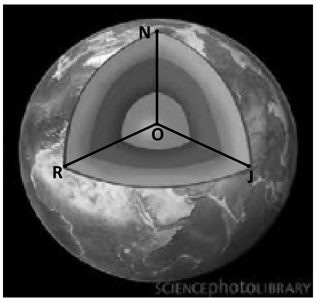
Example 1
Suppose we wanted to find the shortest distance on the earth's surface between Johannesburg and Rio de Janeiro (Do you know where it is?) in Brazil, we would use trigonometry. This method is the same as the one that was used by Greek and Egyptian navigators. Our planet is almost spherical in shape. Long distances are therefore measured using what are known as spherical triangles. Trigonometry plays a central role in this form of measurement.

Solution

If we let *J* be the position of Johannesburg, *R* that of Rio de Janeiro, *N* the North Pole, and *O* the centre of the earth, then, we have a spherical triangle as shown in the figure. The length of the arcs *NR*, *NJ* and *JR* (the distance from Johannesburg to Rio de Janeiro), cannot be measured as straight lines because the earth is round, not flat. You will learn how to solve such problems as you do more of this branch of mathematics. In the beginning, though, you will be learning the basics.



Positions of Rio de Janeiro and Johannesburg



Positions of Johannesburg and Rio de Janeiro on a spherical earth

In this lesson you will:

- think about processes that repeat themselves and periodicity
- write down trigonometric ratios of any angles
- find the sine and cosine functions of angles in triangles
- find the tangent function of angles in triangles

Processes that repeat themselves (periodicity)

Trigonometry is also used to describe mathematical processes that repeat themselves. Can you think of anything that repeats itself in life?

- The hands of a clock (or watch) go round and round and they seem never to stop (unless something is wrong with it).
- Think about the old clocks that had a pendulum. The movement of the pendulum swings backwards and forwards and keeps the clock ticking.
- Think about children playing with swings in the playground.
- Think about machines that do work using rotating wheels.

All these are examples of processes that show repetition. There are many types of motion based on repetition. Have you seen inside a car engine? It has crank shafts and pistons. The movement made by the crank shaft as it rotates drives the car. The mathematical description of the crank shaft's movement is useful to people who design engines.

There are two words that must be discussed. These words are **recurring** and **periodic**. When we say something is recurring we mean that the process repeats itself indefinitely. Any process takes place over time. Repetitions occur over a period of time. Each period is called an interval. These intervals are then called periods of time. That is why we use the word periodic. The movement of the pendulum of a clock is an example of something that is periodic. At certain moments it occupies position A, and it keeps on coming back to the same position. The time taken to move from position A, up to position B, and back to position A, is called a period. There are many other processes that have similar repetitions.

ACTIVITY 1

Take an old tyre of a car or a circular lid from a jar or tin. Mark a position on one side of the tyre, at the bottom. You can use any marker, e.g., a piece of chalk. Let the tyre roll. Watch the chalk mark. It will move up and down as the tyre goes round for each roll. The line of movement that you draw, following the chalk mark, is called the path of the chalk mark. Draw the path of the chalk mark, as it goes up and down. Do this a second time. This time, mark the positions on the ground, whenever the chalk mark reaches its original bottom position. Measure the distances between the marks you made on the ground. What can you say about the distance between the marks you made on the ground?

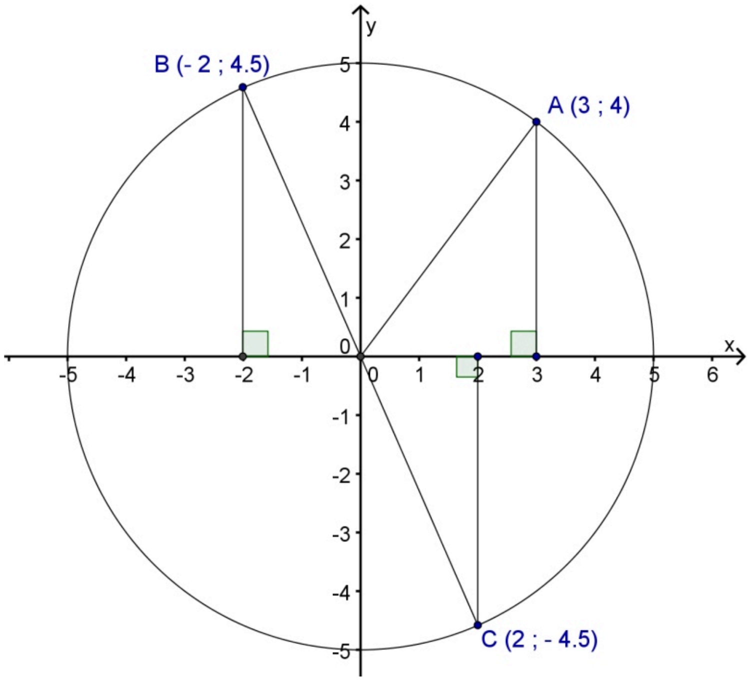
ANSWERS ON PAGE 82

Circular motion

The periodic movement of trigonometry is based on movement in a circle. We need to be able to write the positions of a point as it goes round and round in a circle.

How do we get the positions on the circumference? Well, we use our knowledge of the coordinate system, the same coordinate system we have used to draw linear and quadratic graphs.

Look at this figure.



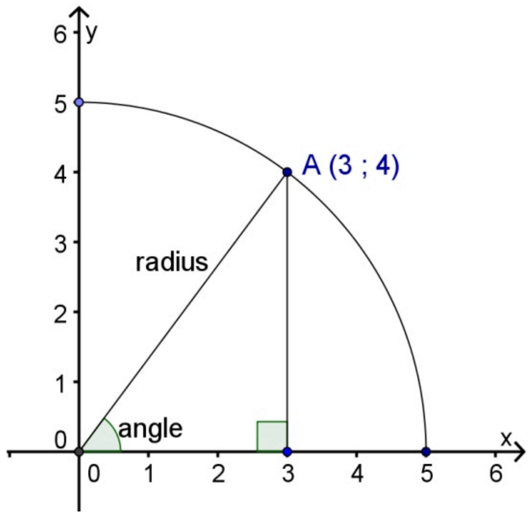
Let us investigate the different positions of a point as it moves around the circumference. Everywhere on the circumference, the position of the point can be given in x and y coordinates. For a start, three of the positions have been identified. When the point is at position A, the coordinates of the point have been identified as $(3;4)$. It moves on to point B, where its coordinates are $(-2;4.5)$. At point C its coordinates are $(2;-4.5)$. This is how we describe the positions of the point anywhere on the circumference. The time taken by the point to move from A, right around the circle and back to the A is called the period.

Note that the lines OA, OB and OC are the radius of the same circle at different positions. As the point revolves around the circle, the angle between the radius and the positive x –axis changes. For example, the angle between OA and the positive x –axis is different from the angle between OB and the positive x –axis. We can describe every position on the circumference using the angle between the radius and the positive x –axis. This allows us to start talking about the x and y values at each angle.

This means that we can describe every position of the point on the circumference using the x –coordinate, the y –coordinate and the angle between the radius and the positive x –axis.

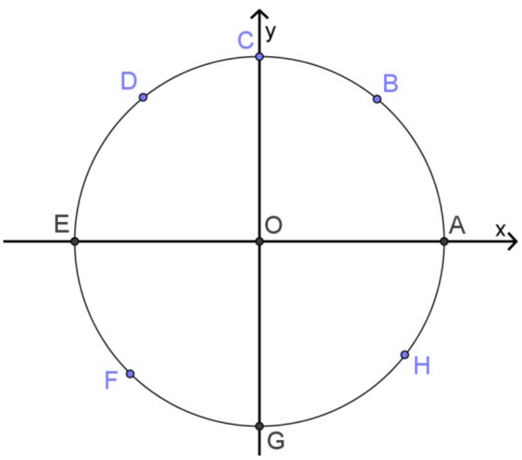
The following diagram shows the different values we use in describing the position of a point on the circumference.

Point A on the circumference has coordinates $(3;4)$. The angle between the radius and the x –axis can be measured. The relationship between these measurements is what trigonometry is based on.



Example 2

Let's look at another example. Have you seen or used a sewing machine?



Let us look at the movement of a wheel of a sewing machine. Let us make a mark on the circumference of the wheel. Suppose the radius of the wheel is 10 cm.

We can draw a system of axes with the centre of the wheel at the position (0; 0). This will allow us to give every position on the wheel in coordinate form.

Every point on the circumference has x and y coordinates. We can choose a few of these positions and mark them. They are shown as the positions A to H here. Each of these positions can be described in terms of its coordinates.

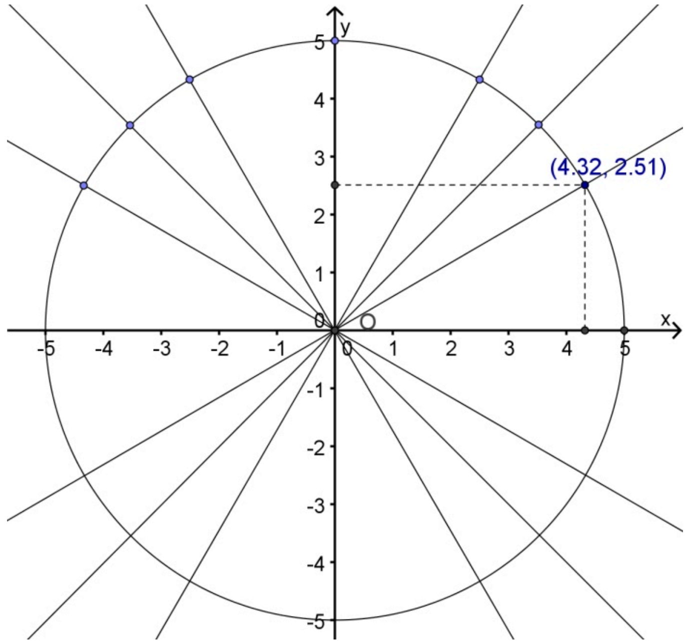
ACTIVITY 2

You will need your mathematical instruments for this activity. Draw a system of axes. Mark out points on both sides of the axes 1 cm apart. 5 marks on each side will be enough. This means that on each axis you will have the following points: –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5. The axes will cross at right angles at 0.

Draw a circle of radius 5 cm on the axes, with the centre at $(0;0)$. Using your protractor, mark out the following angles:

$30^\circ, 45^\circ, 60^\circ, 120^\circ, 135^\circ, 150^\circ, 210^\circ, 225^\circ, 240^\circ, 300^\circ, 315^\circ, 330^\circ$

Draw lines to the origin joining the points you have marked using your protractor. You should have something like this by now:



Now write the coordinates of the points where the lines you drew cut the circumference of the circle. Do you remember how to do that? From each point, draw dotted lines that are perpendicular to the axes. The value of each pair of coordinates will be the value where these perpendicular lines cut the axes. The first point is given as an example.

ANSWERS ON PAGE 82

Now read on. You will compare your results with the example in the following section.

Trigonometric ratios

In order to understand trigonometric functions and their graphs, let us start by looking at functions in general. When we have a relationship (a formula) that can give us known values at any given point, we say we have a function. For example, you have used functions like $y = 2x$. For every given value of x, the independent variable, you can calculate values of y, the dependent variable.

This means that we have to look for a formula that can help us find a value when we have been given one or more other values.

For example, we say speed is a function of time. If you are given the length of time that a certain car takes to travel a certain distance, then you can say what its average speed is.

The same happens in trigonometry. We try to find a formula that will give us the relationship between the coordinates and the angle between the radius and the positive x –axis. Activity 2 has shown you that at given angles the coordinates can be stated. Before going further, try another activity.

ACTIVITY 3

Draw a wheel with radius 10 cm. Use the same angles as in activity 2. Write down the coordinates of the points on the circumference as accurately as possible.

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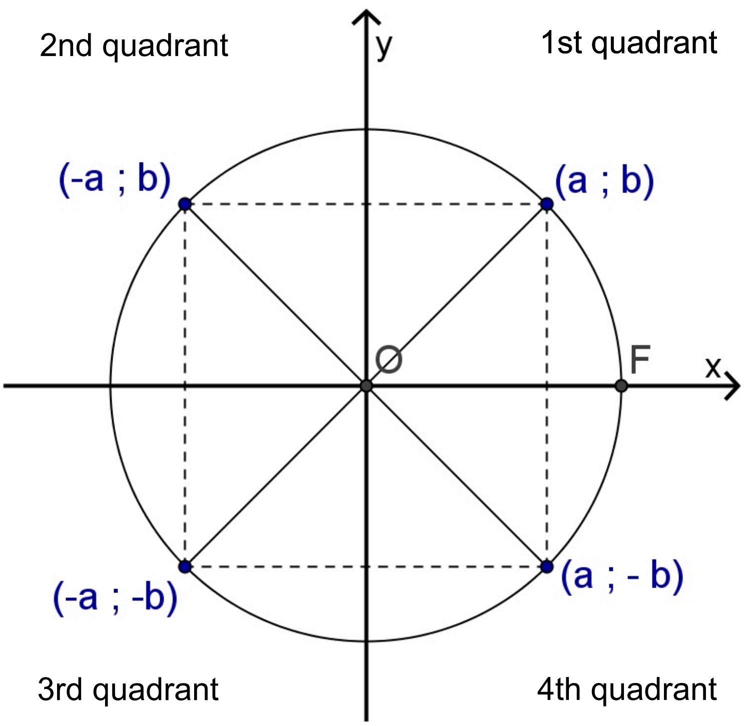
Here is something you may have noticed in the values of the coordinates:



The different areas where the values of coordinates have different signs are called quadrants. There are four quadrants.

The coordinates have changing signs. This depends on the location of the point. If a point is in the first quadrant $\left[0^{\circ}; 90^{\circ}\right]$, the x – and y –coordinates are both positive. In the second quadrant $\left[90^{\circ}; 180^{\circ}\right]$, the x –coordinate is negative, and the y –coordinate is positive. In the third quadrant $\left[180^{\circ}; 270^{\circ}\right]$, both the x and y –coordinates are negative. In the fourth quadrant $\left[270^{\circ}; 360^{\circ}\right]$, the x –coordinate is positive, while the y –coordinate is negative.

ACTIVITY 4

ANSWERS ON PAGE 83

Make a table of the readings from Activities 2 and 3 like the one below. Fill in the values you determined in Activity 2 and 3.

Angles	Activity 2				Activity 3			
	y	$\frac{y}{r}$	x	$\frac{x}{r}$	y	$\frac{y}{r}$	x	$\frac{x}{r}$
30^o								
45^o								
60^o								
90^o								
120^o								
135^o								
etc								

Let us try and find the meaning of the values we have been working with. In all the values in the table there is a relationship between x –values, the radius, and the angle. You may have noticed that there is a relationship between the values of the x –coordinates and the angles. There is also another relationship between the y –coordinates and the angles. You can see this relationship if you look at values in the same row.

In the first row, with the angle equal to 30^o the ratio of $\frac{x}{r}$ is the same for both circles.

Activity 2, $r = 5$, then $\frac{x}{r} = \frac{4.3}{5} = 0.86$

Activity 3, $r = 10$, then $\frac{x}{r} = \frac{8.6}{10} = 0.86$

Also, there is a relationship between the y –values, the radius, and the angle.

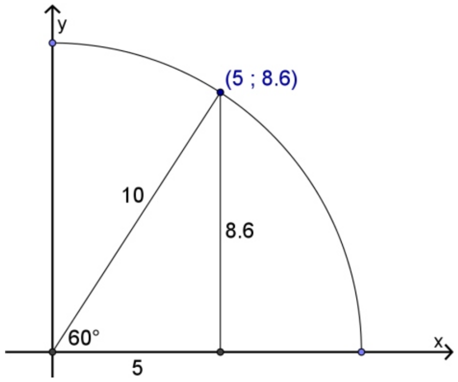
In the first row, with the angle equal to 30^o the ratio $\frac{y}{r}$ is the same for both activities.

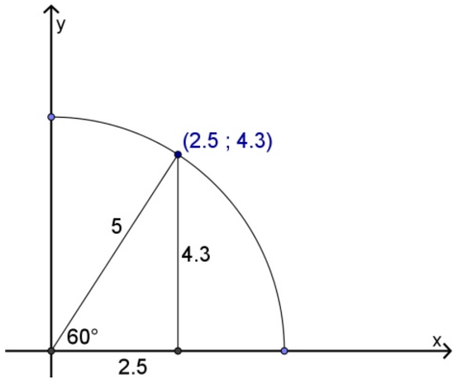
Activity 2, $r = 5$, then $\frac{y}{r} = \frac{2.5}{5} = 0.5$

Activity 3, $r = 10$, then $\frac{y}{r} = \frac{5}{10} = 0.5$

In each row, that is, at each angle, the ratio of $\frac{x}{r}$ is the same, no matter what the radius is. In each row, the ratio $\frac{y}{r}$ is the same, no matter what the radius is. Is there another way of explaining this?

Look at the relevant triangles from activities 2 and 3.





Looking at the two similar triangles above, you can see that $\frac{x}{r}$ in the one triangle is equivalent to $\frac{x}{r}$ in the other triangle. $\frac{y}{r}$ is also equivalent in the two triangles. The diagram shows that no matter what the value of the radius is, if the angle is the same, the ratio will be the same. Let's move on to the next section where we'll learn more about these ratios.

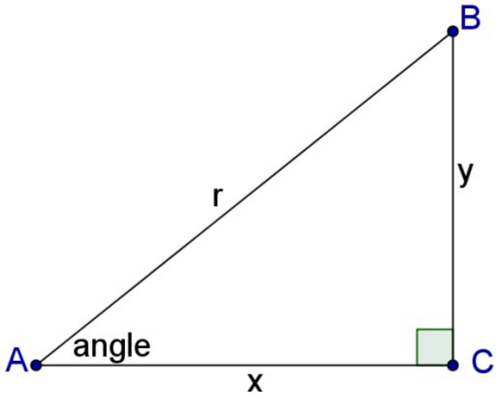
Trigonometric functions

If you look at the previous triangles, the value of the x –coordinate is the same as the length of the side adjacent to the angle of 60°. r is actually the hypotenuse. The value of the y –coordinate is the same as the length of the side opposite the angle of 60°.

> ***adjacent:***
> *lying next to*

In an attempt to make this clearer, look at the diagram below:

AB is the **hypotenuse** which is opposite the right angle. BC is the side **opposite** to angle A. AC is the side **adjacent** to angle A.



As we have seen, no matter how long the hypotenuse is, the value of the trigonometric ratios $\frac{x}{r}$ and $\frac{y}{r}$ will be the same for the same angle. The ratios $\frac{x}{r}$ and $\frac{y}{r}$ are trigonometric ratios. They are the formulas of trigonometric functions. These two basic functions have special names:

> The ratio $\frac{y}{r}$ is known as the **sine of the angle**
>
> The ratio $\frac{x}{r}$ is known as the **cosine of the angle**

Note that the length of the side opposite the given angle is equal to the value of the y –coordinate of the position at the given angle. In the same way, the length of the side that is adjacent to the given angle is equal to the value of the x –coordinate. That is why it is always the case that we talk about the adjacent side and the opposite side, in the following manner:

We abbreviate sine A to sin A and cosine A to cos A

Given a triangle with angle A,

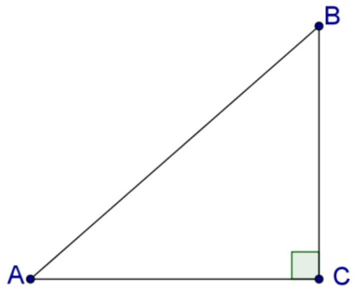
$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

Example 3

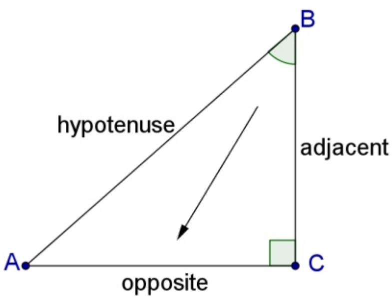
Using the following sketch, write down:

a) the side opposite to angle B

b) the side adjacent to angle B



Solution



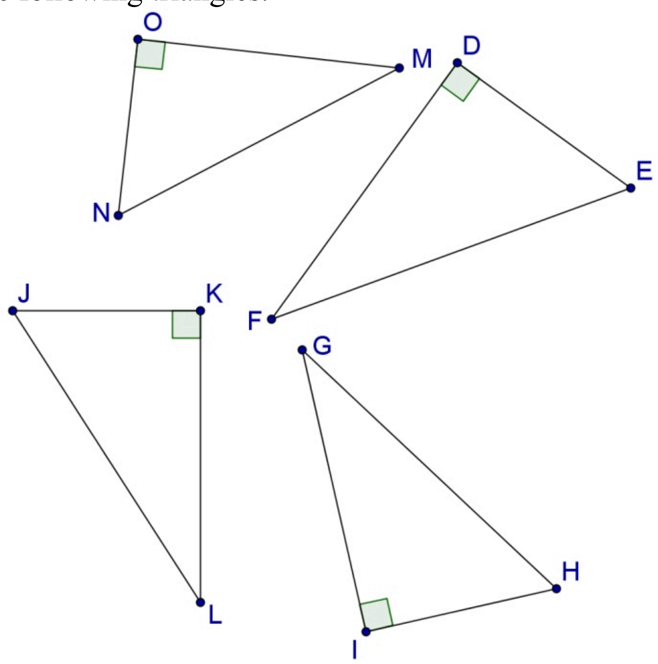
a) The side opposite to angle B is AC

b) The side adjacent to angle B is BC.

NOTE: Although AB is also alongside angle B, side AB is called the hypotenuse, since it is opposite the right angle.

ACTIVITY 5

Look at the following triangles.



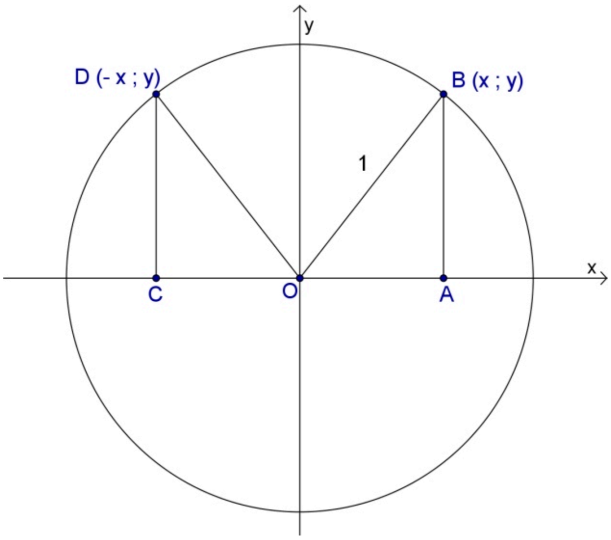
Write down the following sides:

a) The side adjacent to angle E.
b) The side opposite angle F.
c) The side adjacent to angle G.
d) The side adjacent to angle H.
e) The side opposite angle J.
f) The side adjacent to angle L.
g) The side opposite angle N.
h) The side adjacent to angle M.

ANSWERS ON PAGE 84

We have seen that it does not matter how long the radius of the circle is, the ratios $\frac{x}{r}$ and $\frac{y}{r}$ will remain the same.

It is important to point out that often people talk about the sides of a right-angled triangle, and leave out the fact that these are coordinates of the point on a circumference. Let us repeat a few things to make this use of a triangle clear.



In this diagram we have a circle with radius = 1 unit. Remember, a circle with radius = 1 unit is called a unit circle. The position of the point B can be given in $(x;y)$ coordinates. The value of the x –coordinate is the same as the length of OA. The value of the y –coordinate is the same as the length of AB.

Similarly, the position of point D can be given as

$(-x;y) = (-\text{length of OC; length of CD})$

That is why many people say that in triangle OAB

1. $\cos A\hat{O}B = \frac{x}{r} = \frac{AO}{OB}$
2. $\sin A\hat{O}B = \frac{y}{r} = \frac{AB}{OB}$

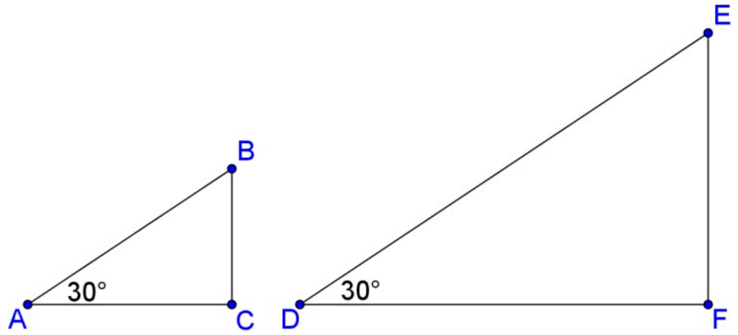
Similarly, in triangle OCD

3. $\cos C\hat{O}D = \frac{x}{r} = \frac{OC}{OD}$ (This will be a negative answer)
4. $\sin C\hat{O}D = \frac{y}{r} = \frac{CD}{OD}$

NOTE: The x –coordinate in point D has magnitude equal to the length of OC. But the coordinate is negative. A trigonometric ratio must always take this into consideration. That is why the ratio in number 3 above will give a negative answer. The hypotenuse will always be positive.

ACTIVITY 6

Draw two right-angled triangles ABC and DEF each with one angle equal to 30º. Make the hypotenuse of one triangle longer than the hypotenuse of the second triangle. (Similar to the drawing below).



a) By measuring the lengths of sides using a ruler, calculate $\cos A$ and $\cos D$.

b) Are the measurements equal?

c) Calculate $\sin A$ and $\sin D$

d) Are the measurements equal?

ANSWERS ON PAGE 84

So far we have dealt with only two basic trigonometric functions. Let us now introduce a third basic function, the tangent function.

The tangent function

Let us look at activity 6 in a slightly different way.

We have not yet compared the opposite and adjacent sides, that is, leaving out the radius altogether. If we look at the 2 triangles drawn in Activity 6, we find that $\frac{BC}{AC} = \frac{EF}{DF}$.

This means that we have a new ratio, $\frac{y}{x}$. Let's do an activity.

ACTIVITY 7

Turn back to Activities 2 and 3 where circles of radii 5 cm and 10 cm were drawn. Calculate the ratios $\frac{y}{x}$ for each angle in each of the activities 2 and 3. The table in Activity 4 may help. You may use a calculator for this purpose. For example:

Angles	Activity 2			Activity 3		
	y	x	$\frac{y}{x}$	y	x	$\frac{y}{x}$
30º	2.5	4.3	0.58	5	8.6	0.58

ANSWERS ON PAGE 84

It is easy to see that we can find a ratio of $\frac{y}{x}$ for each given angle. We also see that we get the same ratio for the same angle, even if the lengths of x and y change. That means we have a formula again. We can therefore talk about a function that gives us certain values at given angles.

This function is called the tangent function. The formula that is commonly used is as follows:

The tangent of a given angle $= \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$

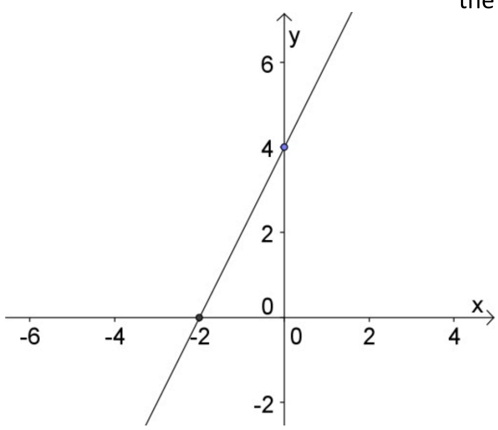
The ratio $\frac{y}{x}$ is known as the **tangent of the angle**

The slope of a function

The word **tangent** is used in a number of ways. A very common example is the slope of a function. The ratio $\frac{y}{x}$ gives an understanding of slope. Let us say we had a function as shown in the figure below.

This function is going through two points $(-2; 0)$ and $(0; 4)$.
From the first point to the second, the change in y –coordinates is 4, while the change in x –coordinates is 2.

We can write the slope of the function as follows: $\frac{\text{the change in y}}{\text{the change in x}}$



If you look carefully at the diagram, you will see that the graph, with the x and y axes, makes a right angled triangle. The above relationship then will give the tangent value of the angle between the graph and the positive x –axis.

This is how a tangent is interpreted in geometry. You can test this in the following activity.

ACTIVITY 8

Draw a system of axes, i.e. the x –axis and the y –axis. Draw the straight line that cuts through the points $(-3; 0)$ and $(0; 3)$. Call the angle between this line and the x –axis angle A.

Measure angle A. Find the tangent of angle A using a calculator. Compare this value with the value of the ratio of the lengths of the opposite and the adjacent sides i.e. $\frac{y}{x}$.

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Therefore the tan of the angle that it makes with the positive x –axis is equal to the slope of the function.

We now have three trigonometric functions, the sine, cosine, and tangent. It is important to note that we can find the tangent by dividing the sine of an angle by the cosine of that angle. This is easy to understand, since

We abbreviate sine A to sin A and cosine A to cos A and tangent A to tan A

$$\cos A = \frac{x}{r}$$

$$\sin A = \frac{y}{r}$$

$$\begin{aligned}\frac{\sin A}{\cos A} &= \frac{y}{r} \div \frac{x}{r} \\ &= \frac{y}{r} \times \frac{r}{x} \\ &= \frac{y}{x} \\ &= \tan A\end{aligned}$$

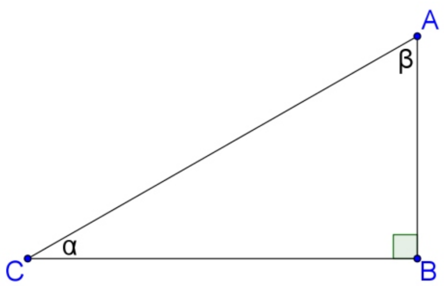
Let us look at a triangle and try to identify these functions. We will write these functions using ratios of the lengths of the sides of this triangle. In doing this, let us introduce Greek letters that are often used to give names to angles.

α - This is a Greek letter which is read **alpha**, pronounced ‘alfa’.
β - This is a Greek letter which is read **beta**.
θ - This is a Greek letter which is read **theta**.

We shall be using these letters. You will also find them in many books of mathematics. Try the following activity.

ACTIVITY 9

1. Using the diagram opposite, write down the following ratios:
   a) $\sin a$
   b) $\cos\alpha$
   c) $\tan\alpha$
   d) $\sin\beta$
   e) $\cos\beta$
   f) $\tan\beta$



2. What is the sum of the two angles, alpha and beta, equal to? What can you say about the sine and cosine of such angles?

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Whenever we have complementary angles, the sine of one is equal to the cosine of another. Look at the triangle ABC.
$\alpha + \beta = 90^o$
$\therefore$ angles α and β are complementary angles.

Now look at the answers in no. 1. It is easy to note the relationship between the sine, cosine and tangent ratios of the two complementary angles. We can summarise the relationships in the following manner:

$$\sin\alpha = \frac{AB}{AC} = \cos\beta$$

$$\sin\beta = \frac{BC}{AC} = \cos\alpha$$

$$\tan\alpha = \frac{AB}{BC} = \frac{1}{\tan\beta}$$

Summary

This lesson has touched a number of ideas in a topic which may be new to you. You have learnt that there are mechanical processes that repeat themselves at particular periodic intervals. In addition you have learnt about circular motion in trigonometry and how you can use your knowledge of coordinates to draw linear and quadratic graphs. You have also looked at trigonometric ratios and functions and learnt about the tangent function and the slope of the function.

CHECKLIST

Are you able to:
- give examples of periodic functions
- write down the sine, cosine and tangent of angles in terms of the sides of given triangles

Remember the following formulae:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

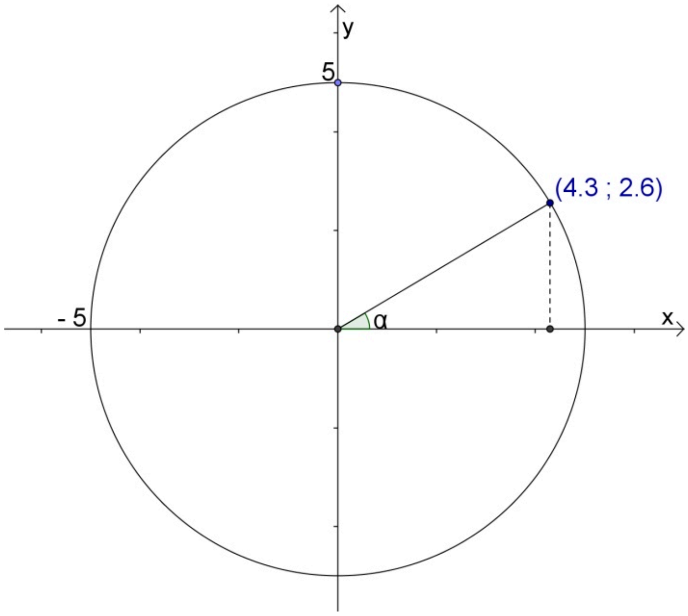
Check yourself in the following exercises.

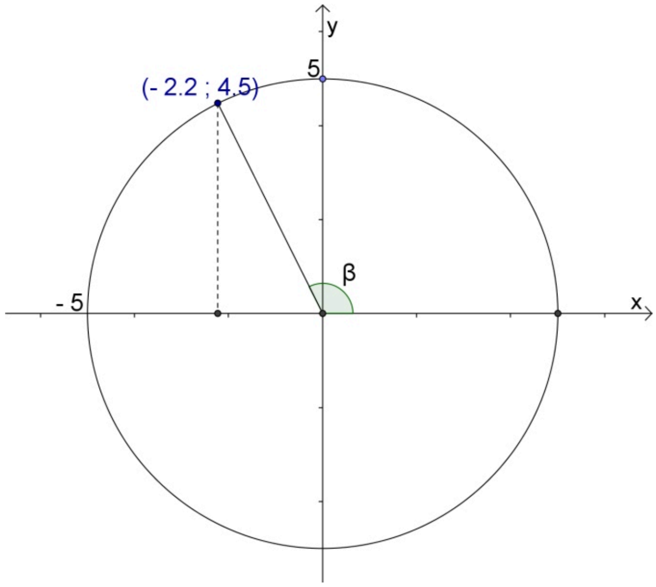
SELF-CHECK EXERCISE

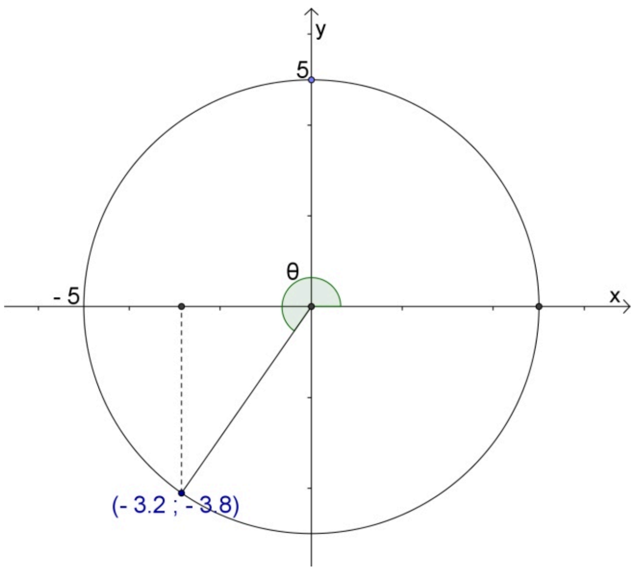
1. Give two examples where periodic movements occur. Why do you say such movements are periodic?

2. Define the basic trigonometric functions sine, cosine and tangent, using x, y and r, and also using the adjacent side, the opposite side, and the hypotenuse.

3. Here is a circle with given points on the circumference. Find the sine, cosine and tangent of angles α, β and θ.







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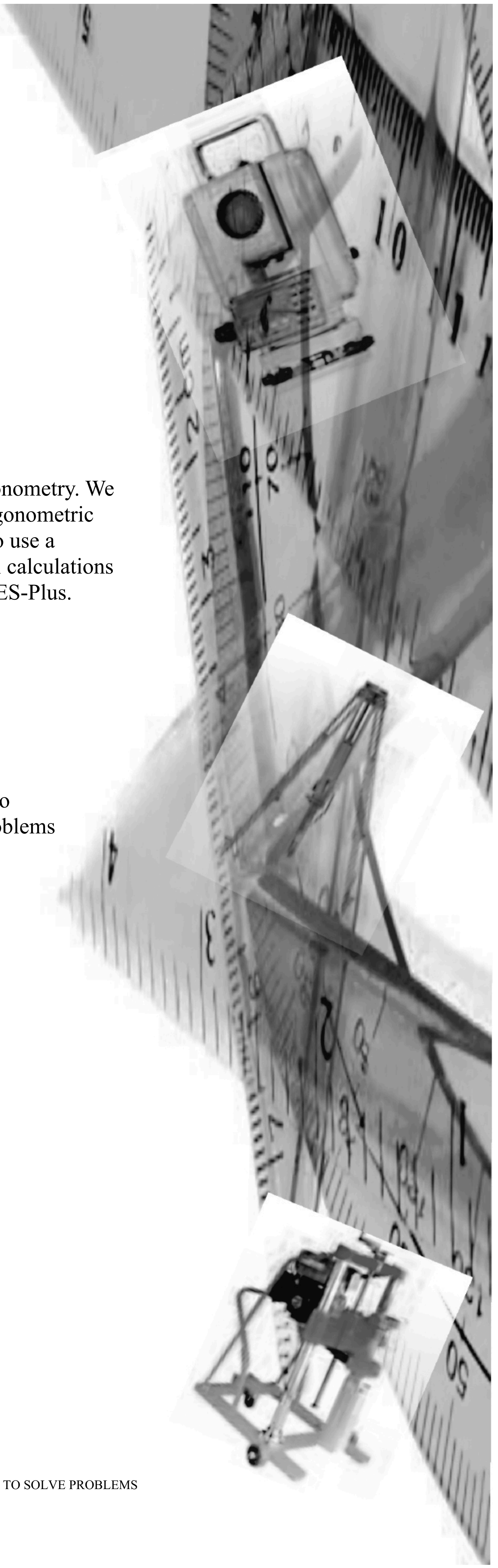
LESSON 4

Using Trigonometry to Solve Problems



About this lesson

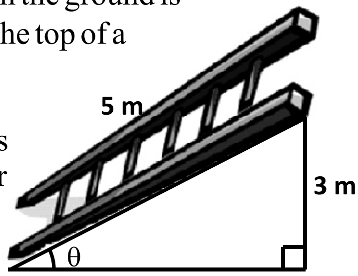
So far we have looked at much of the theory behind trigonometry. We will now take a look at some of its uses involving the trigonometric ratios sine, cosine, and tangent. We will also learn how to use a calculator to evaluate these functions. Please note that all calculations have been done using a Casio Scientific calculator fx-82ES-Plus.

In this lesson you will:

- use the sine ratio in calculations
- use the cosine ratio in calculations
- use tangent ratio in calculations
- decide which to choose, sine, cosine or tangent ratio
- use trigonometric ratios to solve some practical problems (applications)

Sine Ratio

A 5m ladder will slip if the angle it makes with the ground is less than 50°. If we want the ladder to reach the top of a 3m wall, can we calculate the angle?



Before we can try to solve such a problem it is important that we learn some basic calculator operations.

To write the fraction on the calculator you use the $\frac{\square}{\square}$ button.

Example 1

1 Write down sin of these angles correct to 2 decimal places.

a) 30°

b) 80°

2. Find the angles, rounded off to 2 decimal places, given:

a) $\sin A = 0,9$

b) $\sin B = \dfrac{2}{3}$

Solution

1. a) $\sin 30^{\circ} = 0,5$ $\sin \rightarrow 30 \rightarrow =$

b) $\sin 80^{\circ} = 0,98$ $\sin \rightarrow 80 \rightarrow =$

2. a) $\sin A = 0,9$

$\therefore \hat{A} = 64,16^{\circ}$ $\text{shift} \rightarrow \sin \rightarrow 0.9 \rightarrow) \rightarrow =$

b) $\sin B = \dfrac{2}{3}$

$\therefore \hat{B} = 41,81^{\circ}$ $\text{shift} \rightarrow \sin \rightarrow \frac{\square}{\square} \rightarrow 2 \rightarrow \blacktriangledown \rightarrow 3 \rightarrow \blacktriangleright \rightarrow) \rightarrow =$

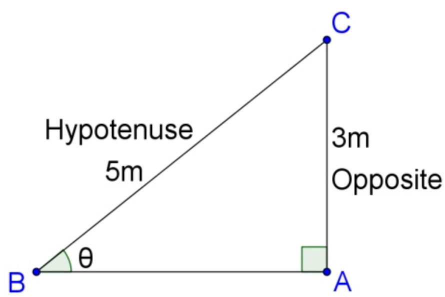
Now we know how to find the sin of an angle and how to find an angle. Let us take another look at the initial problem.

Example 2

A ladder will slip if the angle it makes with the ground is less than 50°. The ladder is 5 metres long and the wall is 3 metres high. Will the ladder slip?

Solution

We must first make a right-angled triangle sketch of the problem.



$$\sin\theta = \frac{opposite}{hypotenuse} \quad (soh)$$

$$\cos\theta = \frac{adjacent}{hypotenuse} \quad (cah)$$

$$\tan\theta = \frac{opposite}{adjacent} \quad (toa)$$

We have been given the value of the opposite side in relation to angle θ and the hypotenuse. Therefore when we choose between *soh* or *cah* or *toa*, we will take the ratio that uses those two sides, the sin ratio.

$$\sin\theta = \frac{o}{h}$$

$$\therefore \sin\theta = \frac{3}{5}$$

$\therefore \theta = 36{,}87^\circ$ shift → sin → $\frac{\square}{\square}$ → 3 → ▼ → 5 → ▶ →) → =

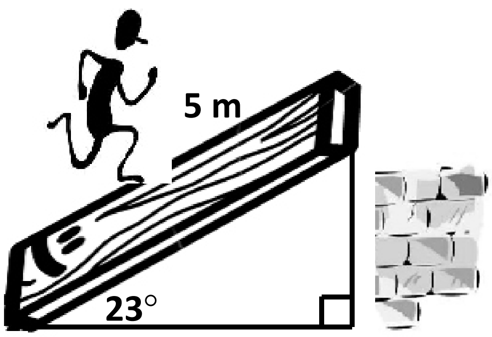
The ladder will slip because the angle of $36{,}87^\circ$ is less than 50°.

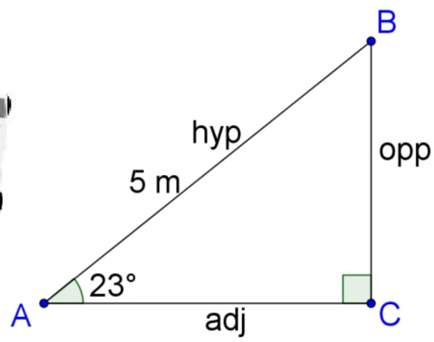
Example 3

Kgomotso runs up a 5 metre plank to the top of a wall. Calculate the height of the wall if the angle between the plank of wood and the ground is 23°.

Solution

Always draw a diagram where possible and fill in all the details given.





Let's call the triangle ABC. You have $\hat{A} = 23^\circ$ and you have the hypotenuse as 5 m but you need to find the height which is opposite the 23°. Since we are looking for the opposite side and we have the hypotenuse, we must choose a ratio that uses those two sides, namely, the sin ratio.

$$\sin A = \frac{opp}{hyp}$$

$\therefore \sin 23^\circ = \frac{h}{5}$ multiply both sides by 5

$$\therefore 5\sin 23^\circ = h$$

$\therefore h = 1{,}95m$ 5 → × → sin → 23 →) → =

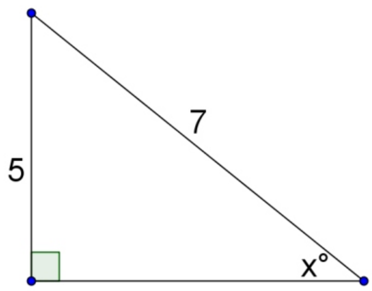
$\therefore$ wall is 1,95 metres high (correct to 2 decimal places)

ACTIVITY 1

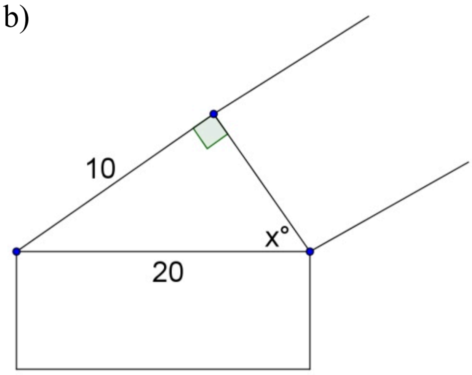
In your answers to the following, give x, the unknown, to the nearest whole number and d correct to 2 decimal places.

Calculate x:

a)

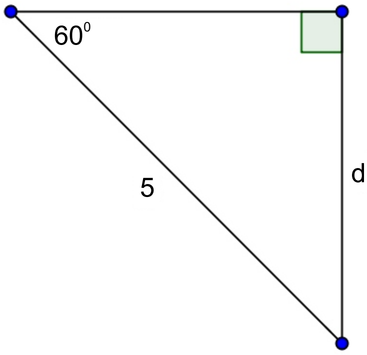


b)

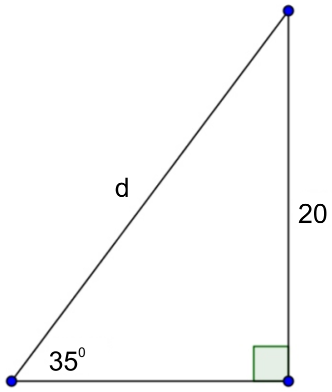


Calculate d:

c)



d)



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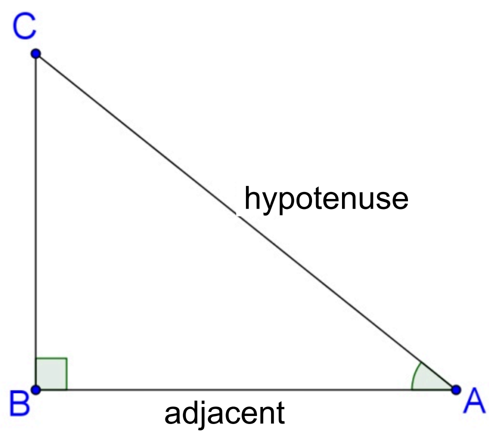
Cosine Ratio

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{AB}{AC}$$

AB is adjacent to A, but opposite C



Example 4

1. Write down the cos of these angles, correct to 2 decimal places where necessary.
   a) 60°
   b) 20°
   c) 0°

2. Find the angles, to the nearest degree, given:
   a) $\cos D = 0,3$
   b) $\cos E = \frac{3}{5}$
   c) $\cos F = 0,84$

Solution

1. a) $\cos 60^{\circ} = 0,5$ cos → 60 → =
   b) $\cos 20^{\circ} = 0,94$
   c) $\cos 0^{\circ} = 1$

2. a) $\cos D = 0,3$
   $\therefore \hat{D} = 73^{\circ}$ shift → cos → 0.3 →) → =

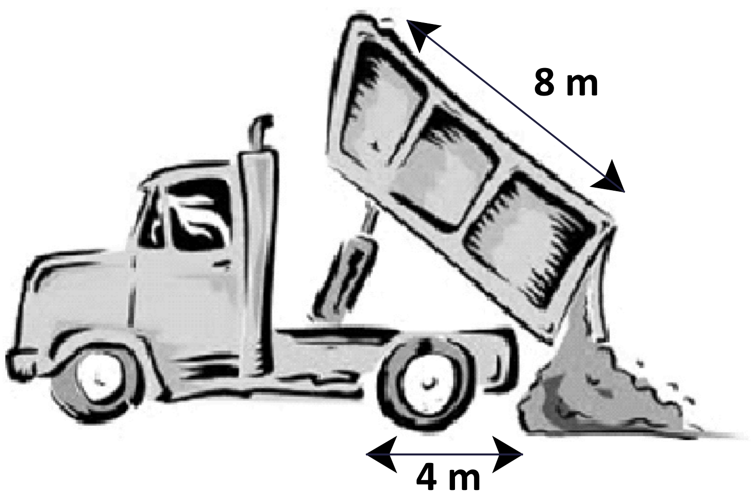
   b) $\cos E = \frac{3}{5}$
   $\therefore \hat{E} = 53^{\circ}$ shift →cos → $\frac{\square}{\square}$ → 3 ▼ → 5 → ▶ →)→ =

   c) $\cos F = 0,84$
   $\therefore \hat{F} = 33^{\circ}$ (to the nearest degree)

We will now look at some of the applications of this rule.

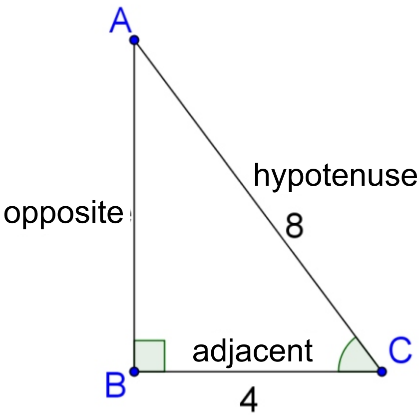
Example 5

Calculate the angle that the truck is tipping at in the following diagram.



Solution

We need to make a right-angled triangle figure to see what exactly we are looking for:



We want to find the size of $\hat{C}$. Identify all the sides of the triangle, remembering that the hypotenuse is always opposite the right angle.

$$\cos C = \frac{adj}{hyp}$$

$$\therefore \cos C = \frac{4}{8}$$

$$\therefore \hat{C} = 60^{\circ}$$

shift → cos → $\frac{\square}{\square}$ → 4 ▼ → 8 → ▶ →) → =

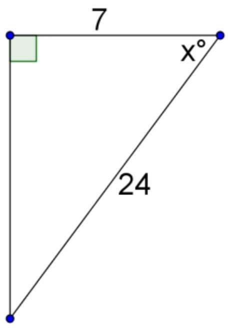
You will recognise much of this work from the sin ratio section so let's move straight into the activity to see how much you can apply.

ACTIVITY 2

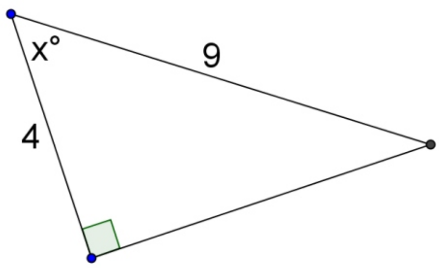
In your answers, give x to the nearest whole number and d correct to 2 decimal places.

Calculate x.

a)

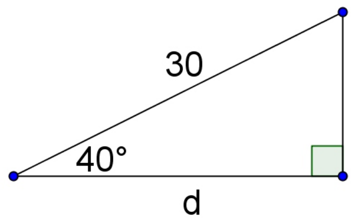


b)

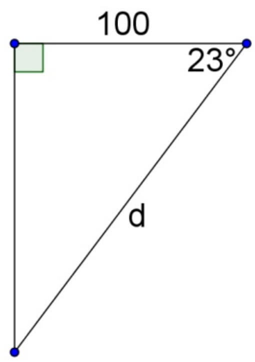


Calculate d

c)

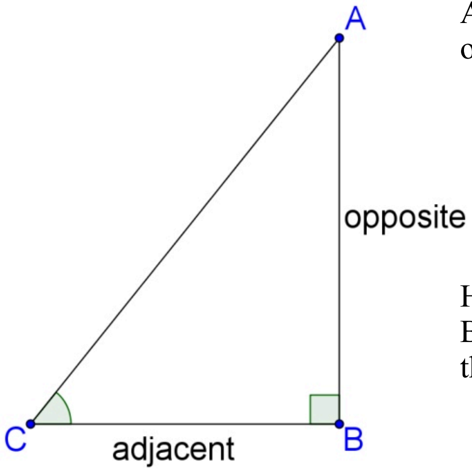


d)



ANSWERS ON PAGE 86

Tangent ratio



Angle C is adjacent to side BC but opposite to side AB

$$\therefore \tan C = \frac{AB}{BC}$$

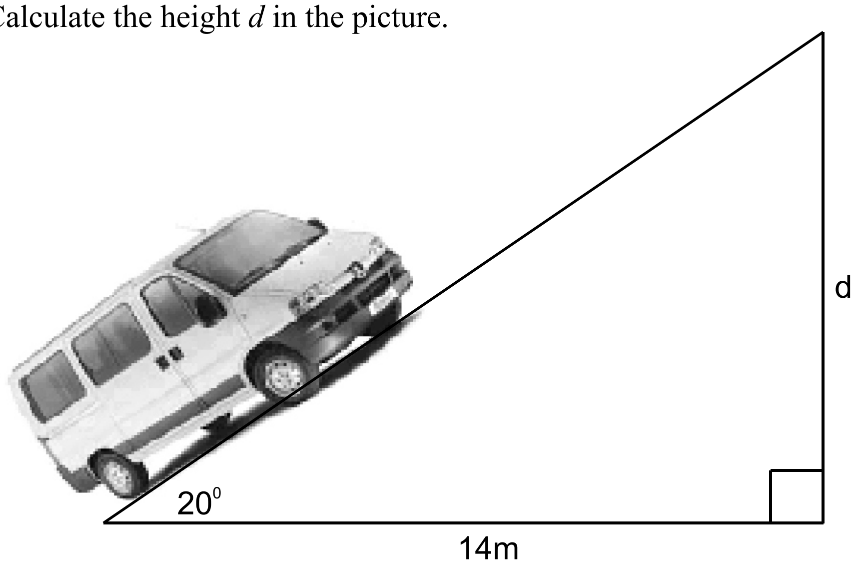
However, if we use angle A, then BC is the opposite side and AB is the adjacent side

$$\therefore \tan A = \frac{BC}{AB}$$

$$\tan\theta = \frac{opposite}{adjacent}$$

Example 6

Calculate the height *d* in the picture.



Solution

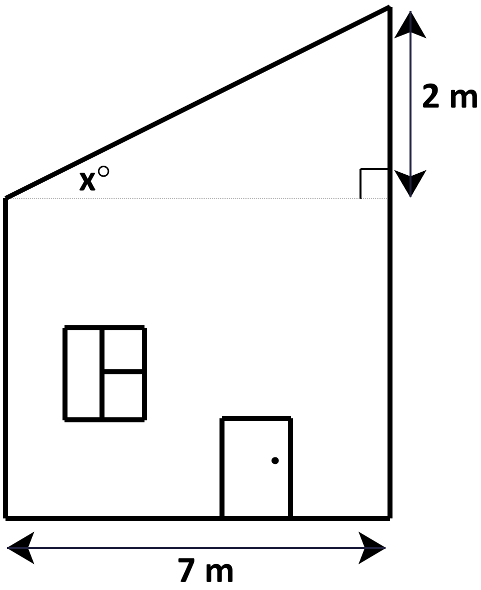
$\tan 20^\circ = \dfrac{opp}{adj}$

$\therefore \tan 20^\circ = \dfrac{d}{14}$

$\therefore 14\tan 20^\circ = d$ multiply both sides by 14

$\therefore d = 5{,}10$ $14 \rightarrow \times \rightarrow \tan \rightarrow 20 \rightarrow) \rightarrow =$
correct to 2 decimal places

ACTIVITY 3

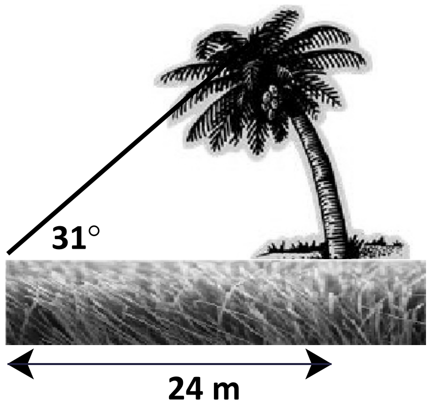


a) Calculate the angle of slope of this roof.

Angle of elevation is the angle between the horizontal and the line to the top of the tree.

ANSWERS ON PAGE 87

b) Calculate the height of the tree, *d*, if the angle of elevation is 31°.

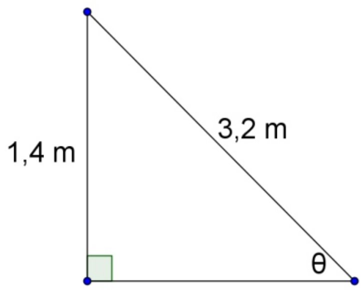


Which to choose, the sine, cosine or tangent ratio?

Up until now, we have dealt with the individual ratios. In this section we will look at various mixed ratio problems to help us to decide which ratio to use.

Example 7

Calculate θ, correct to 2 decimal places.



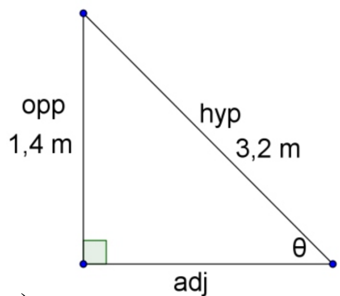
Solution

1. Label the sides in relation to the angle θ.
2. Write down SOH - CAH - TOA.
3. Mark the sides you are given A (adjacent) and/or O (opposite) and/or H (hypotenuse).
4. Use the ratio that has the necessary information.
   In this case, it is the sine ratio.

$$\sin\theta = \frac{opp}{hyp}$$

$$\therefore \sin\theta = \frac{1{,}4}{3{,}2}$$

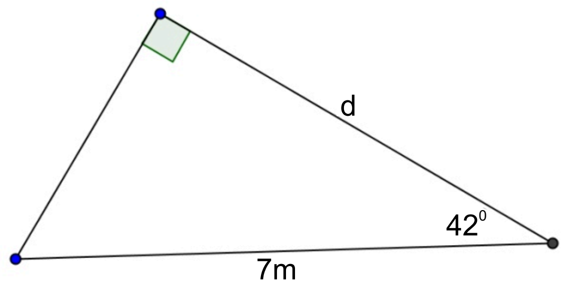
$$\therefore \theta = 25{,}94^{\circ}$$



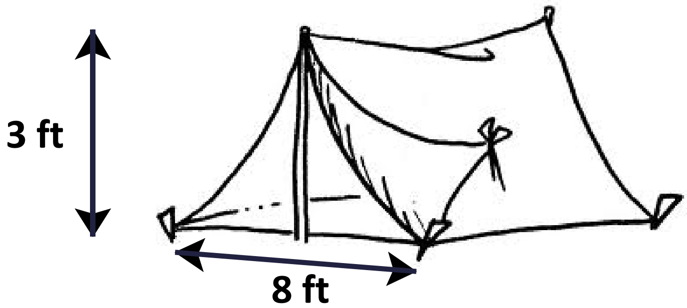
shift → sin → $\frac{\square}{\square}$ → 1.4 ▼→ 3.2 → ▶ →) → =

ACTIVITY 4

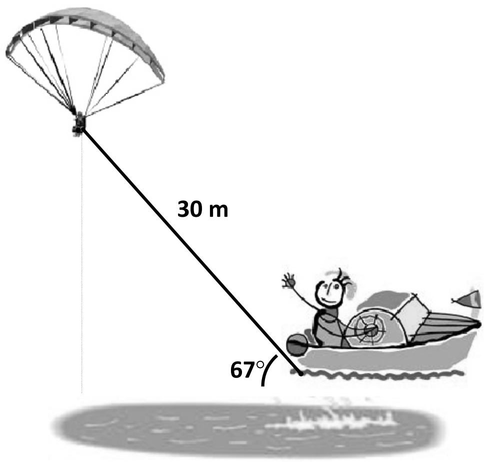
1. Calculate d correct to 2 decimal places.



2. Calculate the angles of slope of the ridge tent, (α) correct to 2 decimal places.



3. Calculate the height above the water of the paraglider, correct to 2 decimal places.



ANSWERS ON PAGE 87

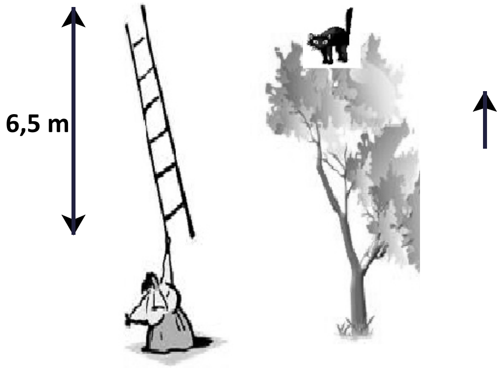
Some more practical applications

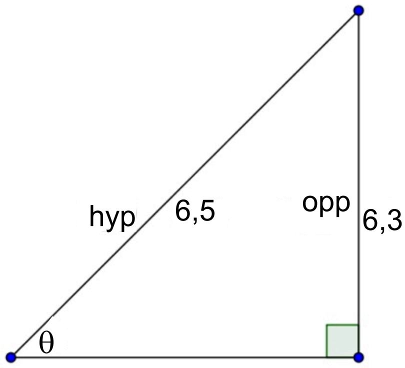
You have seen how trigonometry can be used to solve simple everyday problems. Now let's look at some more examples so that you can practise applying trigonometry.

Example 8

Cat problem: Your cat is trapped on a tree branch 6,3 metres above the ground. Your ladder is only 6,5 metres long. If you place the ladder's tip on the branch, what angle will the ladder make with the ground?

Solution





$$\sin\theta = \frac{6,3}{6,5}$$

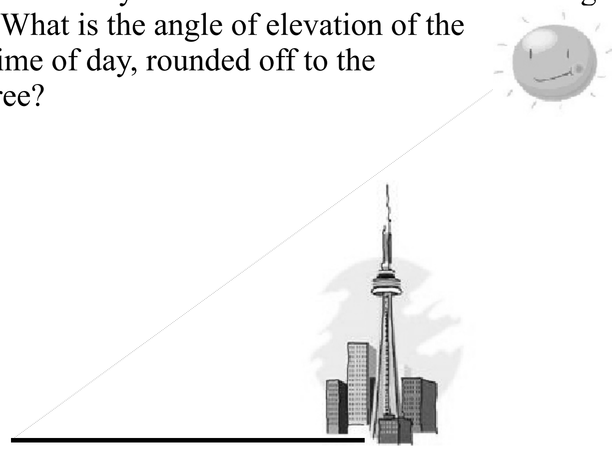
$$\therefore \theta = 75,75^\circ$$

The ladder makes an angle of $75,75^\circ$ with the ground.

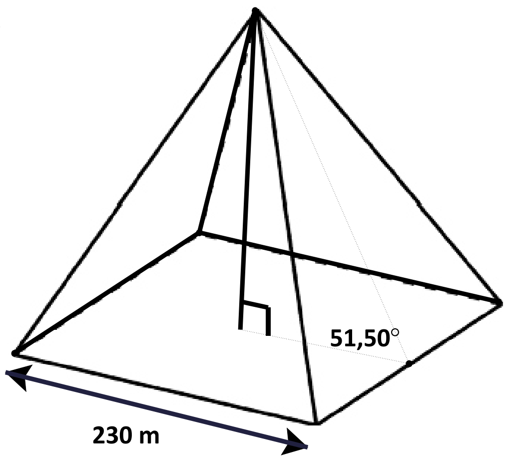
Now try the activity.

ACTIVITY 5

1. The tallest free-standing structure in the world is the 553-metre tall CN tower in Toronto, Canada. Suppose that at a certain time of the day it casts a shadow 1 300 metres long onto the ground. What is the angle of elevation of the sun at that time of day, rounded off to the nearest degree?



2.



The Great Pyramid of Cheops in Egypt has a square base 230 metres on each side.

The faces of the pyramid make an angle of 51,5° with the horizontal. How tall is the pyramid?

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Summary

In this lesson you have learned important facts about the use of trigonometry. You learnt that the sine of an angle = $\frac{\text{opposite}}{\text{hypotenuse}}$; the cosine of an angle = $\frac{\text{adjacent}}{\text{hypotenuse}}$ and the tangent of an angle = $\frac{\text{opposite}}{\text{adjacent}}$.

You learnt that when trying to decide which ratio to use, it is important to put all the information on a right-angled triangle drawing and to mark off what you have and need.

Do the self-check exercise to test your understanding of the lesson. If you have problems with any of the questions, go back to the section of the lesson and re-read the explanation and examples. Then try again.

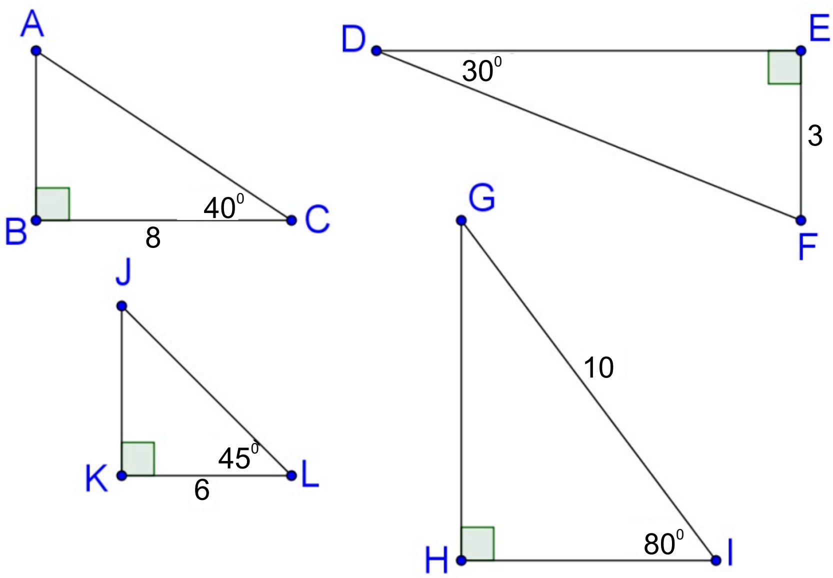
CHECKLIST

Are you able to:

- use sine, cosine and tangent ratios in calculations to solve simple triangular problems
- use a calculator to evaluate sine, cosine or tangent ratios

SELF-CHECK EXERCISE

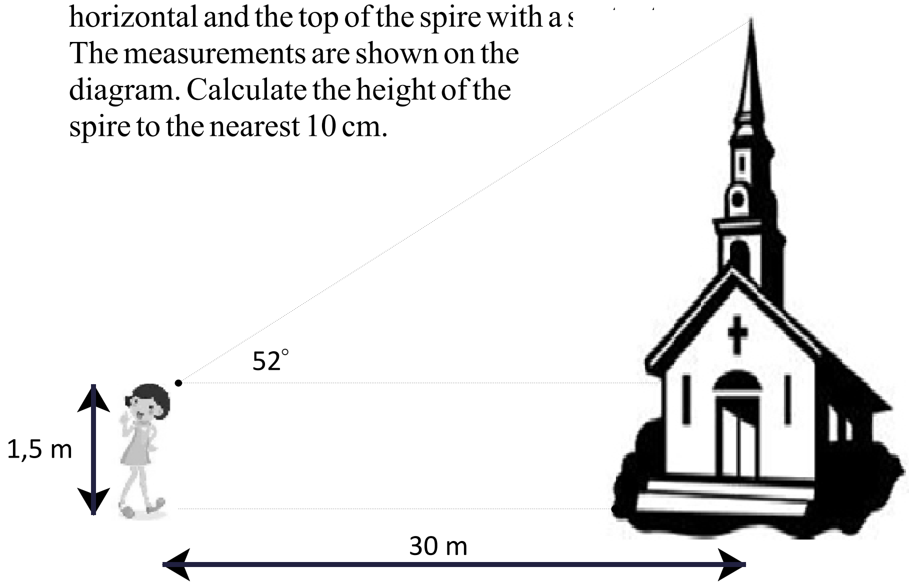
1 In the following diagrams we have right angled triangles:



Use trigonometric ratios to find the lengths of the following sides. Express all your answers to two decimal places.

a)	AC	b)	DE	c)	DF
d)	GH	e)	HI	f)	JK
g)	JL				

2 A student is trying to calculate the height of the church spire in order to win the 'Guess the heigh' competition at school. She measures out a distance of 30 m from the church. From there she measures the angle between the horizontal and the top of the spire with a s
The measurements are shown on the diagram. Calculate the height of the spire to the nearest 10 cm.



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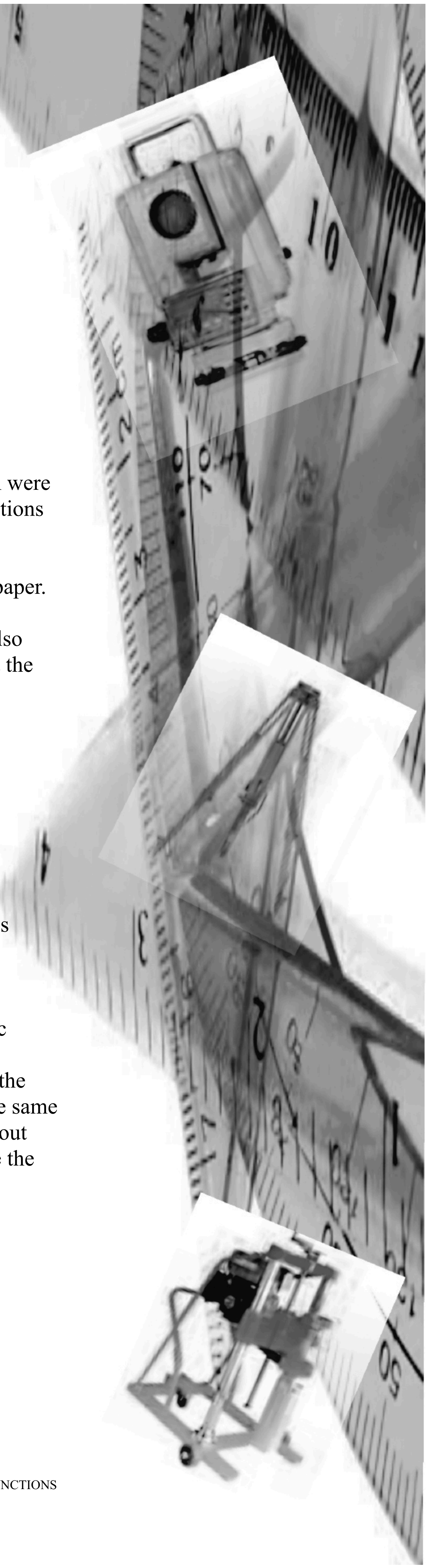
LESSON 5

Graphs of Trigonometric Functions



About this lesson

In Lesson 3, trigonometric ratios as functions of periodic motion were discussed. In this lesson we will discuss the graphs of these functions further.

You will need your pencil, drawing instruments, calculator and paper. You already know how to draw straight-line graphs and curves because you have drawn linear graphs and parabolas. You will also have to remember what you learned in the previous lesson about the values of trigonometric functions at different angles.

In this lesson you will:

- draw the graphs of sine, cosine and tangent functions
- learn about negative angles
- calculate the sign of each of the trigonometric functions in the four quadrants
- calculate the sign of trigonometric functions of large angles consisting of many revolutions

You have used your calculator to find the values of trigonometric ratios at any angle between 0° and 360°. This is one revolution, a movement from 0° around the whole circumference and back to the starting position. That is why the value of a function at 360° is the same as the value of that function at 0°. So far we have been talking about the angles between 0° and 360°. In this lesson we will introduce the idea of negative angles.

But first, let's look at the graphs of trigonometric ratios.

What the graphs look like

To get a general idea about the shape of trigonometric functions, we can use point plotting. That is, choose a series of x –values and substitute them into the desired function in order to get the corresponding y –values. We will then have a number of coordinates which we can plot on the Cartesian plane.

Example

Draw the graph of $y = \sin x$

Solution

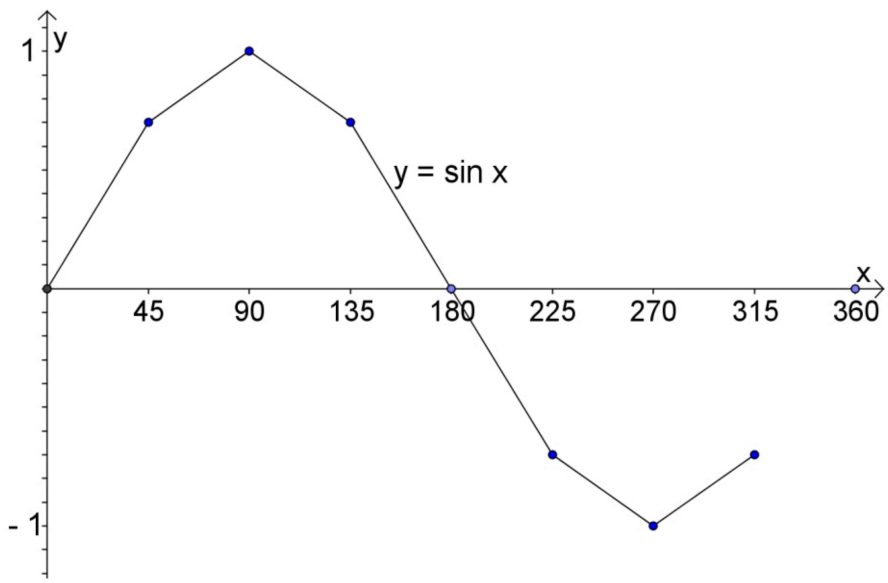
Begin by drawing up a table.

x	0^o	45^o	90^o	135^o	180^o	225^o	270^o	315^o	360^o
$y = \sin x$									

To calculate the value of y when y=sin x, you can use your calculator.

x	45^o	90^o	135^o	180^o	225^o	270^o	315^o	360^o
$y = \sin x$	0,7	1	0,7	0	$-0,7$	-1	$-0,7$	0

Having completed the table you now need to plot the points on the Cartesian plane and join them to make a graph. This is what it looks like:



But something is not right. The shape of the graph is not satisfactory. This is because of the insufficient number of coordinates given. In straight lines, two points are needed to draw a line. In the drawing of curves, more points are necessary, especially turning points. Do you remember about the smoothness of the curve of a parabola? Do the following activity to remind yourself.

ACTIVITY 1

The graph above is not correct. It has a mistake that many people make. Explain what is wrong with the graph and draw a better one. You may want to plot some extra points.

Notice that the graph does not have sharp corners. It is a smooth curve. This is something that you will have to remember.

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Graphs of sine and cosine functions

The previous graph is a sine curve. It is a graph of values of the sine of angles from 0^o to 360^o. Now draw the graph of another trigonometric function.

ACTIVITY 2

Draw the graph of $y = \cos x$, where x stands for all angles between 0^o and 360^o. Don't forget the table of values first. Use a calculator to help you.

ANSWERS ON PAGE 89

The difference between the graphs of $y = \sin x$ and $y = \cos x$ is that the same y - values are given at different values of x. For example:

$\sin x = 0$ when $x = 0^o$
$\cos x = 0$ when $x = 90^o$
$\sin x = 1$ when $x = 90^o$
$\cos x = 1$ when $x = 0^o$

The values of sine and cosine suggest a certain relationship between the two functions. The relationship between the graphs of sine and cosine can be explained in terms of shifts. Can you remember shifts of graphs?

Let us look at a shift of $y = \sin x$ along the x –axis 90^o to the left.
Try the following activity.

ACTIVITY 3

On a piece of tracing paper, draw the graph of $y = \sin x$ for x between 0^o and 360^o. When finished, put the paper on top of the graph you drew in Activity 2. Make sure the axes and the origin $(0;0)$ coincide. Shift (slide) the paper along the x –axis until 90^o on the tracing paper is on top of 0^o of the drawing of Activity 2. What do you see?

This means, for all angles

$$\cos x = \sin\left(x + 90^o\right)$$

The relationship can be explained this way: the cosine curve is equal to the sine curve translated 90^o to the left.

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Negative angles, quadrants and signs of trigonometric ratios

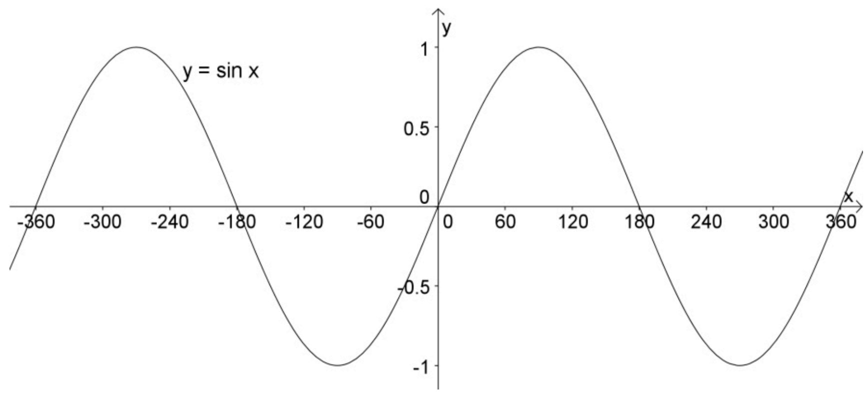
Example

Draw the graph of $y = \sin x$ for x between -360^{o} and 360^{o}. Use your calculator to get the y –coordinates of points.

Solution

You may use your calculator to get all the coordinates. The following is a possible table of values:

x^{o}	-360^{o}	-270^{o}	-180^{o}	-135^{o}	-90^{o}	-45^{o}	0^{o}	45^{o}	90^{o}	135^{o}	180^{o}	270^{o}	360^{o}
$y = \sin x$	0	1	0	−0,7	−1	−0,7	0	0,7	1	0,7	0	−1	0

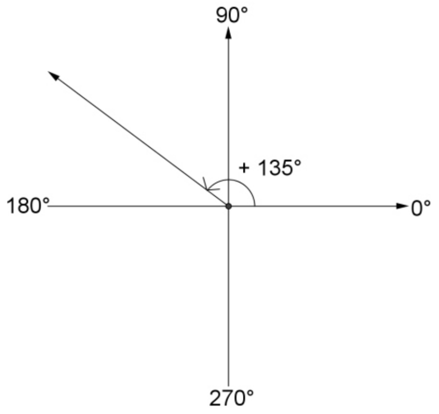


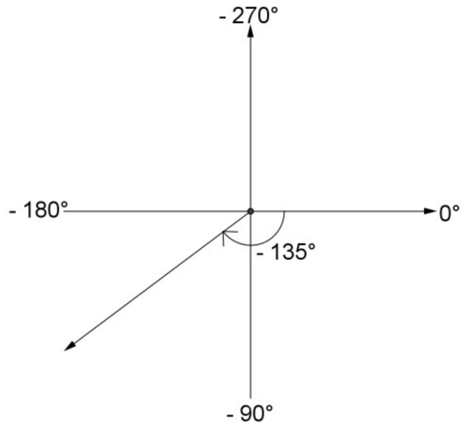
It is easy to draw these graphs. All you need is enough practice. From the graph above you can see that values of trigonometric ratios may be positive or negative without depending on whether the angle is positive or negative. Let's discuss this idea of negative angles further.

Signs that are used before numbers usually relate to direction. You have been using axes or number lines in drawing graphs. The horizontal axis, for instance, has 0 in the middle, with negative numbers decreasing to the left, and the positive numbers increasing to the right. The vertical axis is divided in the same way, with positive numbers increasing upwards above 0 and negative numbers decreasing downwards.

The idea of sign, linked to the direction of movement, also applies in trigonometry. You may have noticed that we have been measuring angles as increasing values from the positive x –axis, moving in a counter-clockwise or anti-clockwise direction. These are all positive angles.

When the direction changes, that is, when we have a clockwise movement around the circumference, we have negative angles. This is the idea of negative angles: angles that show clockwise motion. The following diagrams will show this better.





Positive angles show movement in an anti-clockwise direction.

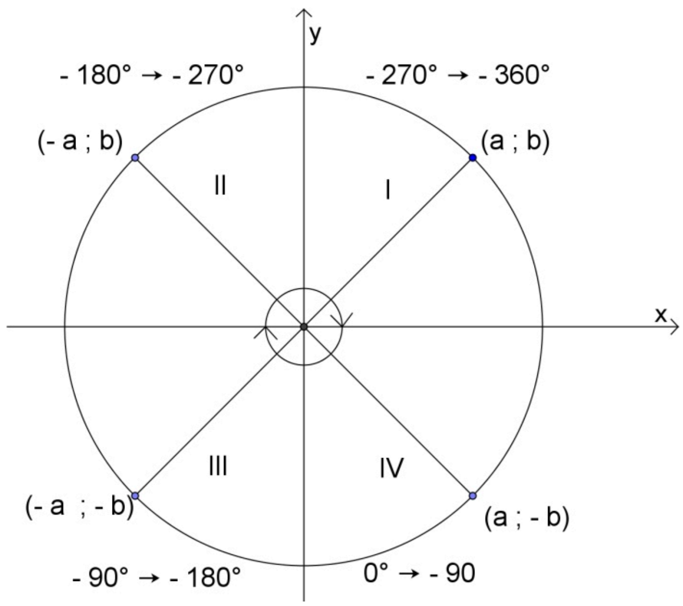
Negative angles show movement in a clockwise direction.

Quadrants

You may already have noticed that the negative angles do not play a special role in the sign of the value of the function. The most important thing affecting the sign of the functions is which quadrant the angle ends in. Always ask yourself this question: In which quadrant is the angle? Do you remember what a quadrant is?

In Lesson 3 we discussed how the system of axes divides the page into four quadrants. We said the first quadrant is from 0^o to 90^o, and so on. What about negative angles? In which quadrant is -60^o?

Look at the following diagram.



Signs of trigonometric ratios

The sign of the ratio is also not affected by the sign of the angle. For example, the tangent of an angle in the first quadrant will always be positive, no matter what the sign of the given angle is.

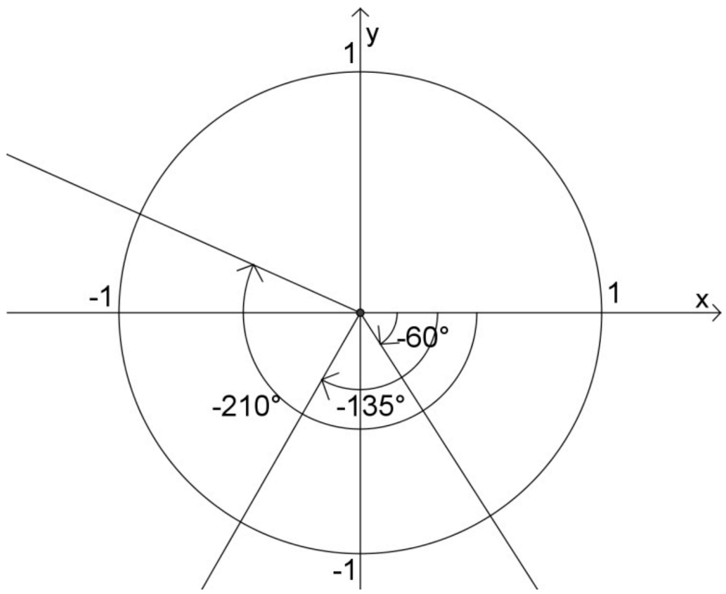
Look at this example:

$\sin(-45^\circ) = -0,707$, negative because -45° is in the fourth quadrant. Values of sine are negative in the fourth quadrant, where we have a negative *y* value divided by *r*, which is always positive, giving a negative value.

$\sin(-225^\circ) = 0,707$, positive because -225° is in the second quadrant, and values of sine are positive in the second quadrant, where y –coordinates are positive. You will learn more about positive and negative ratios in different quadrants in Unit 3.

Example

The following diagram shows a few examples of negative angles. The most important thing to remember is the quadrant in which the angle ends. This informs you what the sign of the ratio is. There is little to remember if you use a calculator. The calculator will always give you the value you want with its sign.



ACTIVITY 4

Do the following activity.

$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

Use your calculator to find the sine, cosine and tangent of 30°, –30°, 60°, –60°, 150°, –150°, 225°, –225°.

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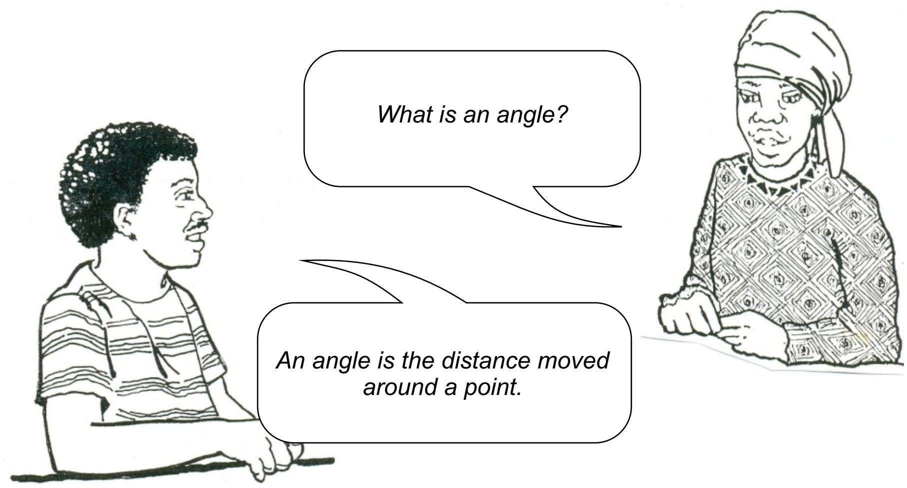
Having used your calculator, you may be able to generalise on signs of trigonometric functions.

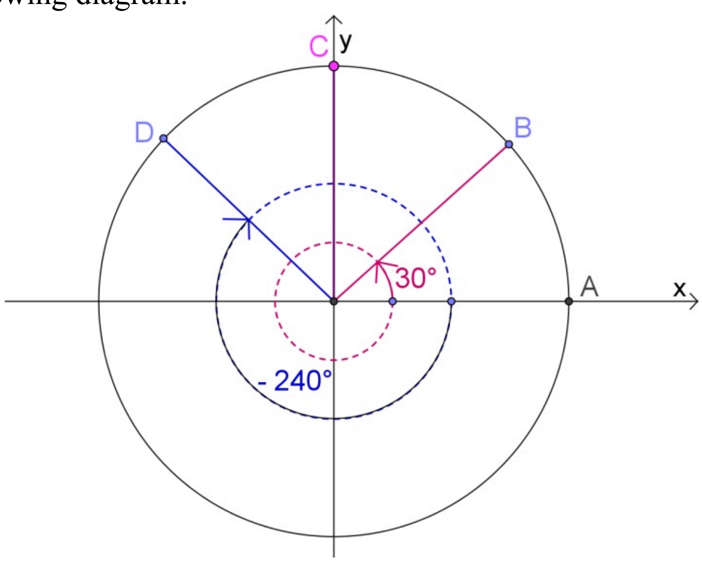
It is also possible to work with very big positive or negative angles. Angles may be bigger than 360°. Our first discussion of trigonometry explained it as a study of functions based on repetitive movement. When a wheel moves around, it may move clockwise or anti-clockwise. That is why we need something that can work in both directions. Moreover, the wheel can turn around many times. That is why we need to work with very big angles. We will discuss this in the following section.

Angles on a number line. Many revolutions



How does this definition explain angles in trigonometry? Let us again look at the movement of a point on the circumference of a circle. The line joining the centre to the point on the circumference is the radius. The angle measured is between the radius and the positive x –axis.

Look at the following diagram.



What happens if the point keeps on moving?

The answer to this question is: Add more degrees to 360^o. For example, if the point moves right round the circle and passes position A moving on to position C, then it will move $360^o + 90^o = 450^o$. That is how we name very big angles.

This idea of movement around the circumference allows us to explain movement that continues for a very long time. That is how the movement of a wheel of a car or a machine can be described. For example, a movement around the circumference 3630^o means that a point starting from the 0^o position moved round the circle ten times and continued until it reached B i.e. the 30^o position.

This use of 'big' values of angles can describe wheel movement.

The interesting part is that the values of the function just repeat themselves. This is when we start talking about periodicity. Try the following activity.

ACTIVITY 5

ANSWERS ON PAGE 91

Draw the graph of the cosine function given angles from -900^o to 900^o. Use your calculator.

Domain

The function drawn in Activity 5, $y = \cos x$ is drawn for all the angles of x between -900^o and 900^o. The y –values depend on the x –values. To calculate any y –value we substitute an x –value into the equation. The collection 'set' of all possible x input values in a function is called the **domain** of the function.

In the graph drawn in Activity 5, we could describe the domain as follows:

x is any angle from -900^o and 900^o.
We write this mathematically as $x \in \left[-900^o ;900^o\right]$

The y –values that are produced by the function can be described by the statement $-1 \leq y \leq 1$. These values are called the **range** of the function $y = \cos x$.

You have so far drawn graphs of two functions, the sine and cosine functions. Let us proceed to the third one, the tangent function.

The graph of the tangent function

In Lesson 3 you read that the tangent function can be given by the formula $\tan\theta = \dfrac{opp}{adj}$

This can also be expressed in two other ways, i.e: $\tan\theta = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$

Your calculator can help you find the tangent of a given angle. Try the following activity.

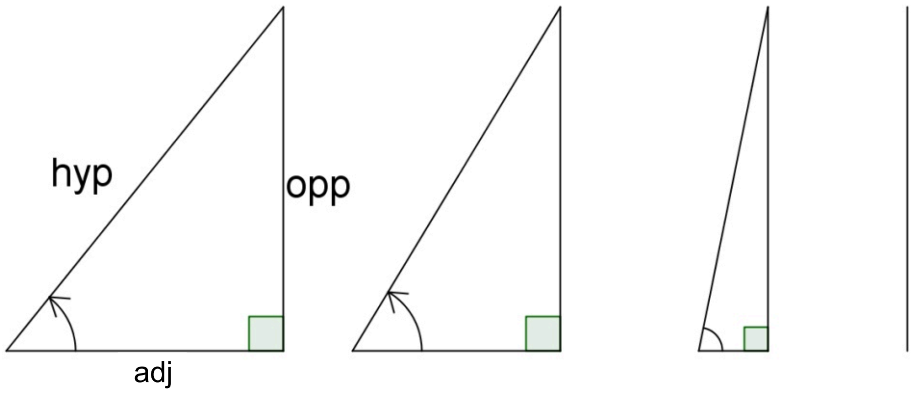
ACTIVITY 6

Draw the table of values for the graph of $y = \tan x$ where x is between -90^{o} and 90^{o}, $x \in [-90^{o}, 90^{o}]$ Use the following values:

x^{o}	-90^{o}	-75^{o}	-60^{o}	-45^{o}	-30^{o}	-15^{o}	0^{o}	15^{o}	30^{o}	45^{o}	60^{o}	75^{o}	90^{o}
$y = \tan x$													

ANSWERS ON PAGE 91

Note that the tangent does not exist when the angle is -90^{o} or 90^{o} .



In the rotation in a circle the 90^{o} angle gives a single vertical line on the y –axis.

If we think of this line as a triangle having a hypotenuse, opposite and adjacent, then the length of the adjacent side must be 0.

So if $\tan x = \frac{opp}{adj}$ or $\tan x = \frac{\sin x}{\cos x}$

then $\tan 90^{o} = \frac{opp}{0}$ or $\tan 90^{o} = \frac{\sin x}{\cos x} = \frac{1}{0}$

But division by 0 is undefined.

So we say that $\tan 90^{o}$ is **undefined**.

Similarly, tan (-90^{o}) is **undefined**.

To find out what happens to the tangent graph between 75° and 90°, try the following activity:

ACTIVITY 7

Complete the following table of values for $y = \tan x$.

x°	75°	80°	85°	86°	87°	88°	89°	$89{,}5^\circ$	$89{,}8^\circ$	$89{,}9^\circ$	90°
$y = \tan x$											

ANSWERS ON PAGE 91

So you see, as the angle gets closer and closer to 90°, the tangent of the angle increases more and more quickly. At 90° itself, there is no tangent.

ACTIVITY 8

Draw the graph of $y = \tan x$ for $x \in \left[-90^\circ ; 90^\circ\right]$.

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In Lesson 1 we looked at asymptotes. This is a line or curve that shows a set of points which the graph gets closer and closer to, without touching or crossing it.

Tangent is not only undefined at –90° and 90°, but is undefined whenever the cosine values are equal to zero. The tangent graph goes through zero whenever sin of the angle equals zero.

For example:

$\tan 180^\circ = 0 \quad \therefore \quad \tan 180^\circ = \dfrac{\sin 180^\circ}{\cos 180^\circ}$

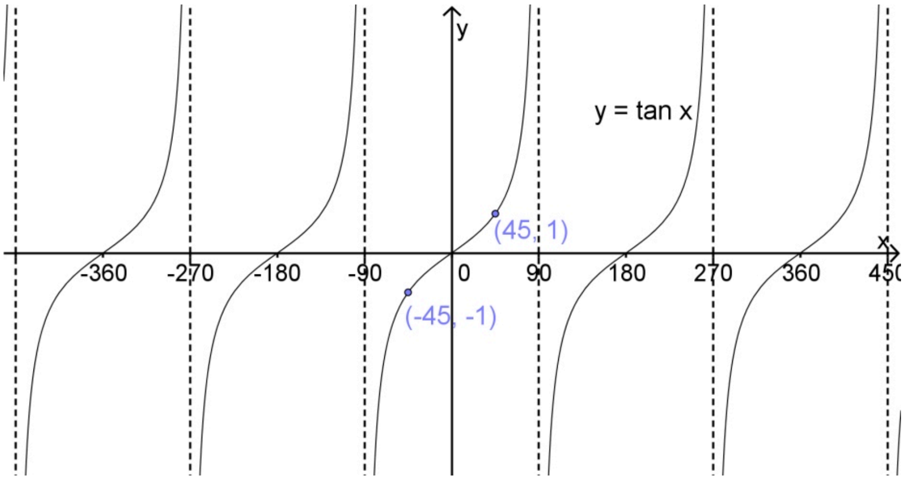
We know that $\sin 180^\circ = 0$ and $\cos 180^\circ = 1$

$\therefore \tan 180^\circ = \dfrac{0}{1} = 0$

Whereas, $\tan 270^\circ$ is undefined

$\tan 270^\circ = \dfrac{\sin 270^\circ}{\cos 270^\circ} = \dfrac{-1}{0}$ which is **undefined**

At all angles where the tangent of such angles does not exist, we have asymptotes. The following diagram of the graph of $y = \tan x$ will demonstrate.



Summary

In this lesson you learnt how to draw graphs of trigonometric functions. You should be able to draw three trigonometric graphs now: the sine, cosine and tangent graphs.

You also learnt about the four quadrants that we use in trigonometry. You saw that changes occur in the signs of the values of trigonometric functions for angles in these quadrants.

Finally, you learnt about negative angles, and values of trigonometric functions of negative angles.

CHECKLIST

Are you able to:

- draw the graphs of the sine, cosine and tangent functions for positive and negative angles
- decide whether the trigonometric ratio of an angle is positive or negative, depending on which quadrant the angle is in.

Test yourself in the following exercise.

SELF-CHECK EXERCISE

1. Draw graphs of each of the following pairs of graphs for $x \in \left[0^\circ; 360^\circ\right]$
   a) $y = \cos x$ and $y = 2\cos x$
   b) $y = \sin x$ and $y = \sin 2x$

   Hint: In $y = 2\cos x$, the value of $\cos x$ has been multiplied by 2 but in $y = \sin 2x$, the angle is multiplied by 2 before you find the ratio. Your table of values could be as follows:

x^{o}	0^{o}	30^{o}	60^{o}	90^{o}	...
$y = \cos x$					
$y = 2\cos x$					

x^{o}	0^{o}	45^{o}	90^{o}	135^{o}	...
$2x$					
$y = \sin 2x$					

2. Draw a graph of $y = \sin A$ for all angles between -450^{o} and 450^{o}.

3. Sketch the following functions for $x \in \left[-180^{o}; 360^{o}\right]$ using your knowledge of transformations.

   a) $y = \sin x + 1$
   b) $y = \cos\left(x + 30^{o}\right)$
   c) $y = 2\cos x + 2$

ANSWERS ON PAGE 97

4. Why is it not possible to work out the value of $\tan 450^{o}$?

Feedback to Activities

Lesson 1

Activity 1

x	$-0,8$	$-0,6$	$-0,4$	$-0,2$	$0,2$	$0,4$	$0,6$	$0,8$
y	-5	$-6,67$	-10	-20	20	10	$6,67$	5

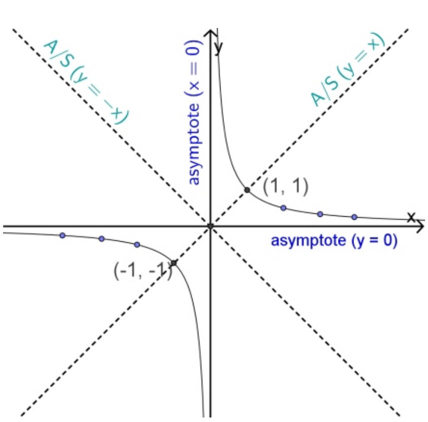
Activity 2

x	-12	-10	-8	-6	6	8	10	12
y	$-0,34$	$-0,4$	$-0,5$	$-0,67$	$0,67$	$0,5$	$0,4$	0.34

Activity 3

x	-4	-3	-2	-1	0	1	2	3	4
$y=\frac{1}{x}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	/	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$y=-\frac{1}{x}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	/	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$
$y=\frac{6}{x}$	$-\frac{3}{2}$	-2	-3	-6	/	6	3	2	$\frac{3}{2}$
$y=-\frac{2}{x}$	$\frac{1}{2}$	$\frac{2}{3}$	1	2	/	-2	-1	$-\frac{2}{3}$	$-\frac{1}{2}$

1.



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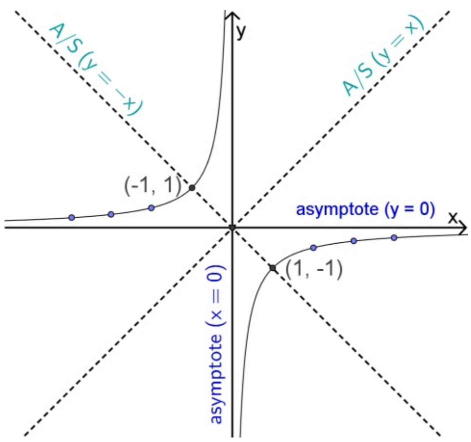

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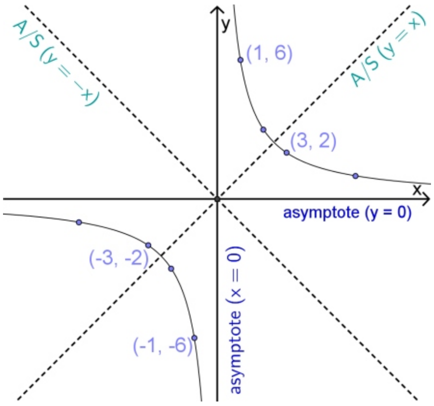

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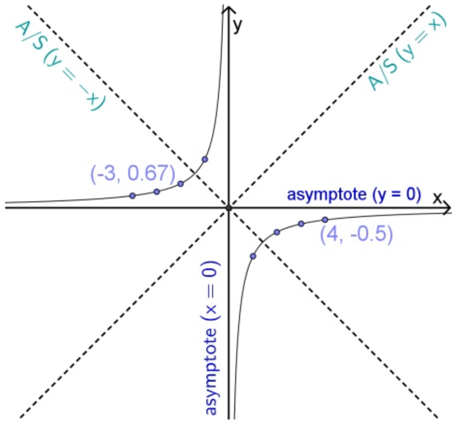
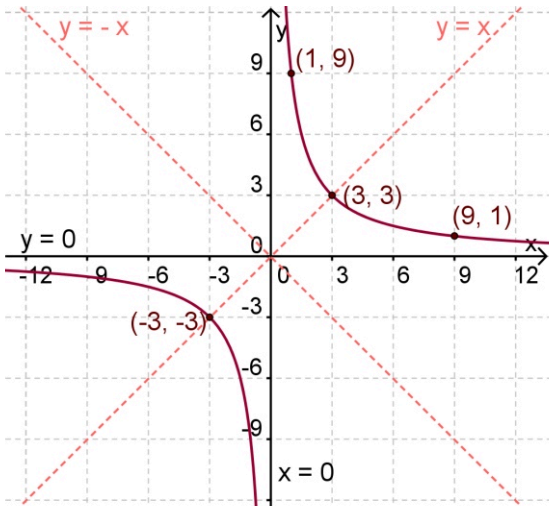
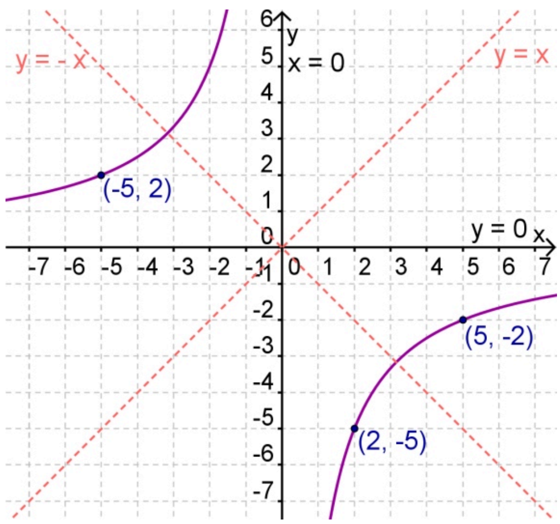


Activity 4

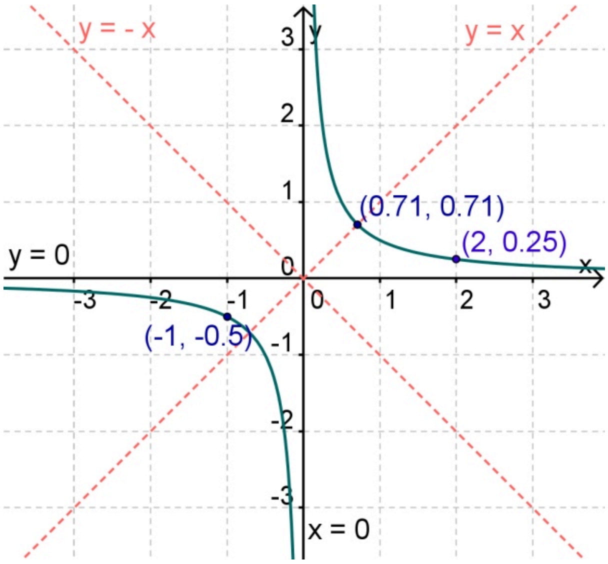
1. quadrants 1 and 3
   points $(3;3)$, $(1;9)$ and $(9;1)$ are on the curve
   two asymptotes, two axes of symmetry



2. quadrants 1 and 3
   points $(-5;2)$, $(5;-2)$ and $(2;-5)$
   two asymptotes, two axes of symmetry

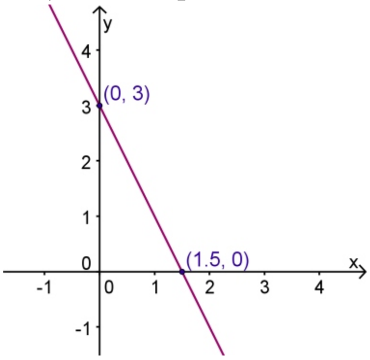


3.



Activity 5

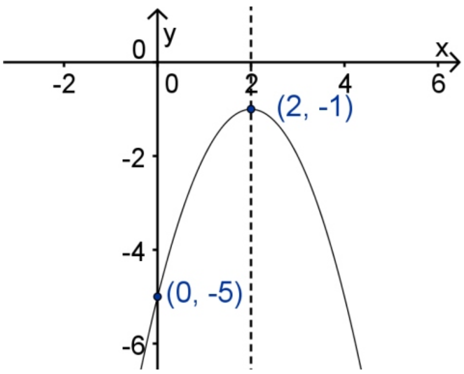
1. $y = -2x + 3$
Simplest way to draw a straight line graph is to calculate the 2 intercepts. To find the x –intercept, we make $y = 0$ and to find the y –intercept we make $x = 0$.



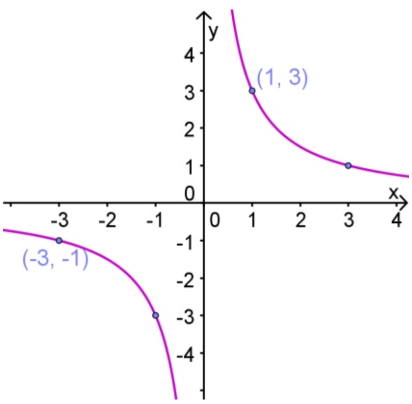
This graph has no restrictions on either it's x –values or y –values.
Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}$

2. $y = -(x-2)^2 - 1$

Domain: $x \in \mathbb{R}$
Range: $y \in (-\infty ; -1]$



3.



$y = \frac{3}{x}$

Domain: $x \in \mathbb{R}$, $x \neq 0$
Range: $y \in \mathbb{R}$, $y \neq 0$

Activity 6

1. $y = \frac{9}{x} - 3$

Let $y = 0$

$\therefore 3 = \frac{9}{x}$

$\therefore 3x = 9$

$\therefore x = 3$

$\therefore$ the x –intercept is 3.

Points:

$(3;3) \to (3;0)$

$(-3;-3) \to (-3;-6)$

Asymptotes:

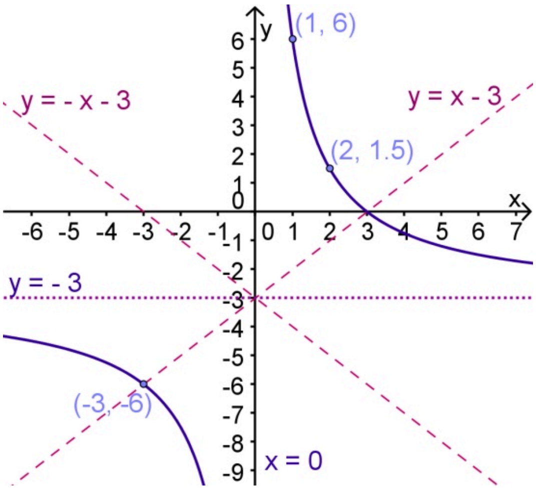
$y = 0 \to y = -3$

$x = 0$ stays the same

Axes of Symmetry:

$y = x \to y = x - 3$

$y = -x \to y = -x - 3$



2. $y = -\frac{10}{x} + 5$

Let $y = 0$ $\quad \therefore -5 = -\frac{10}{x}$

$\therefore -5x = -10$

$\therefore x = 2$

Points: $(-5;2) \to (-5;7)$

$(5;-2) \to (5;3)$

Asymptotes:

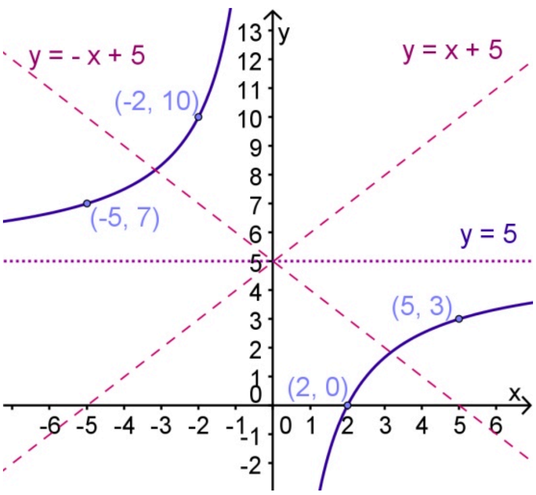
$y = 0 \to y = 5$

$x = 0$ stays the same

Axes of Symmetry:

$y = x \to y = x + 5$

$y = -x \to y = -x + 5$



Activity 7

1. $y = -\dfrac{4}{x+2}$

y –intercept: Let $x = 0$

$\therefore y = -\dfrac{4}{0+2} = -2$

Asymptote:

$x = 0 \rightarrow x = -2$

$y = 0$ stays the same

Axes of Symmetry:

$y = x \rightarrow y = x + 2$

$y = -x \rightarrow y = -(x+2)$

$\therefore y = -x - 2$

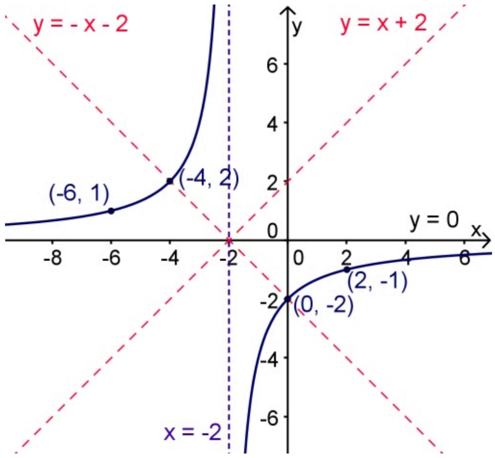
Points on the curve:

$(2;-2) \rightarrow (0;-2)$

$(-2;2) \rightarrow (-4;2)$

$(4;-1) \rightarrow (2;-1)$

$(-4;1) \rightarrow (-6;1)$



2. $y = \dfrac{1}{x-2}$

y –intercept: Let $x = 0$

$\therefore y = \dfrac{1}{0-2} = -\dfrac{1}{2}$

Asymptote:

$x = 0 \rightarrow x = 2$

$y = 0$ stays the same

Axes of Symmetry:

$y = x \rightarrow y = x - 2$

$y = -x \rightarrow y = -(x-2)$

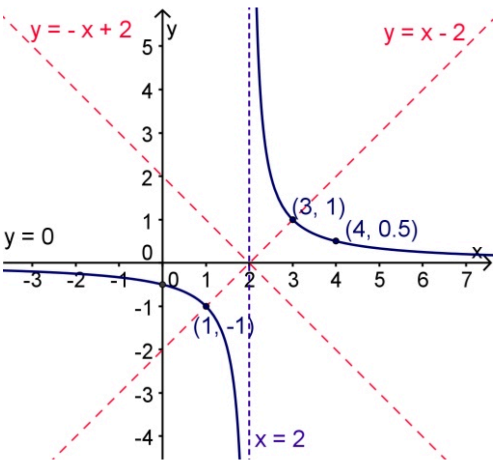
$\therefore y = -x + 2$

Points on the curve:

$(1;1) \rightarrow (3;1)$

$(-1;-1) \rightarrow (1;-1)$

$\left(2;\dfrac{1}{2}\right) \rightarrow \left(4;\dfrac{1}{2}\right)$



Activity 8

1. $y=-\dfrac{1}{x-2}+1$

x –intercept:

$-1=-\dfrac{1}{x-2}$

$\therefore -x+2=-1$

$\therefore -x=-3$

$\therefore x=3$

y –intercept:

$y=-\dfrac{1}{0-2}+1$

$\therefore y=\dfrac{3}{2}$

Asymptote:

$x=0 \rightarrow x=2$

$y=0 \rightarrow y=1$

Points on the curve:

$(-1;1)\rightarrow(1;2)$

$\left(2;-\dfrac{1}{2}\right)\rightarrow\left(4;\dfrac{1}{2}\right)$

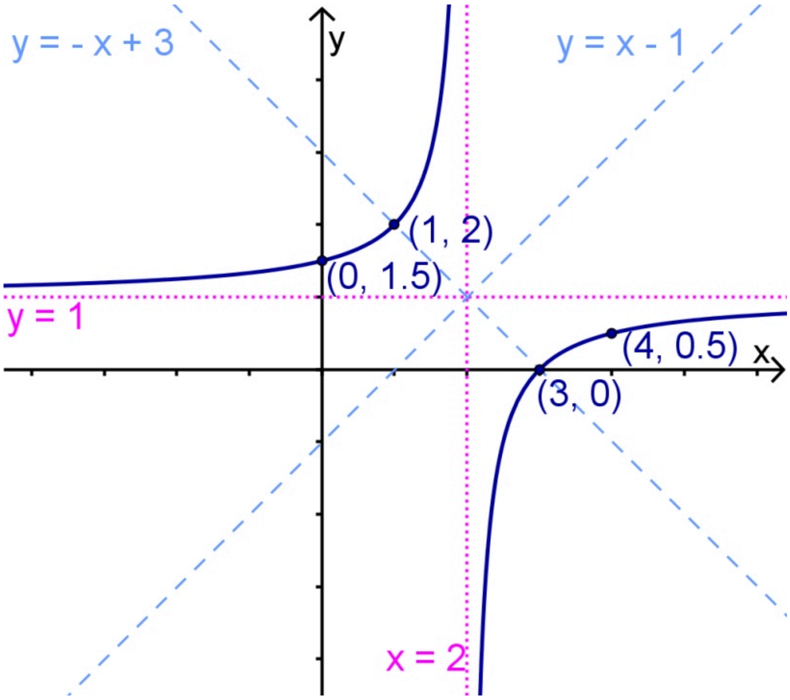
Axes of symmetry:

$y=x \rightarrow y=(x-2)+1$

$\therefore y=x-1$

$y=-x \rightarrow y=-(x-2)+1$

$\therefore =-x+3$



2. $y=\dfrac{2}{x+4}-1$

x –intercept:

$1=\dfrac{2}{x+4}$

$\therefore x+4=2$

$\therefore x=-2$

y –intercept:

$y=\dfrac{2}{0+4}-1$

$\therefore y=-\dfrac{1}{2}$

Asymptote:

$x=0 \rightarrow x=-4$

$y=0 \rightarrow y=-1$

Points on the curve:

$(-1;1)\rightarrow(1;2)$

$\left(2;-\dfrac{1}{2}\right)\rightarrow\left(4;\dfrac{1}{2}\right)$

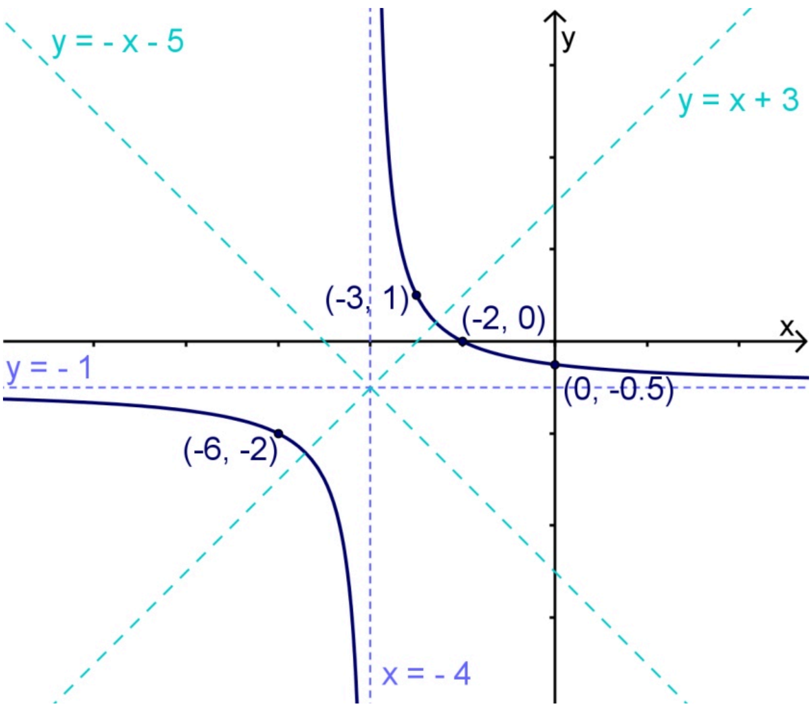
Axes of symmetry:

$y=x \rightarrow y=(x+4)-1$

$\therefore y=x+3$

$y=-x \rightarrow y=-(x+4)-1$

$\therefore =-x-5$



Lesson 2

Activity 1

1. a) Equation of line is $y = 3$
Equation of shaded region is $y < 3$ (dotted line)

b) Equation of line is $x = -2$
Equation of shaded region is $x \geq -2$ (solid line)

c) Equation of line is $y = 3x + 3$
$\therefore$ equation of shaded region is $y \geq 3x + 3$

If in doubt, take a point in the shaded region like $(-2;0)$ and substitute it into the line $y = 3x + 3$

$$LHS = y \qquad RHS = 3(-2) + 3$$
$$= 0 \qquad = -3$$

The LHS > RHS,$\therefore$ $y \geq 3x + 3$

If the line is not given in standard form, you will have to write it in standard form before you describe the shaded region.

d) $y \geq -\frac{2}{3}x - 2$

e) We need to rewrite the line $5x + 2y = -10$ in standard form

$\therefore 5x + 2y = -10$

$2y = -5x - 10$

$y = -\frac{5}{2}x - 5$

Only now can we describe the shaded region to be $y \leq -\frac{5}{2}x - 5$.

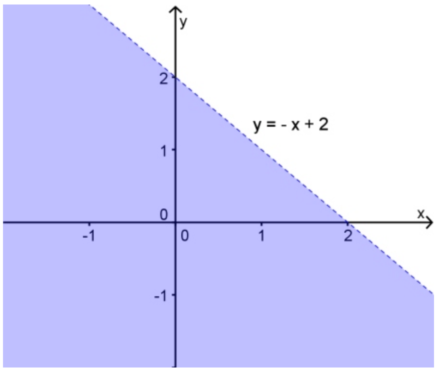
f) In this line we are given the intercepts but not the equation of the line, we will need to find the equation of the line first.
Given the points $(2;0)$ and $(0;-2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-2 - 0}{0 - 2}$$
$$= 1$$

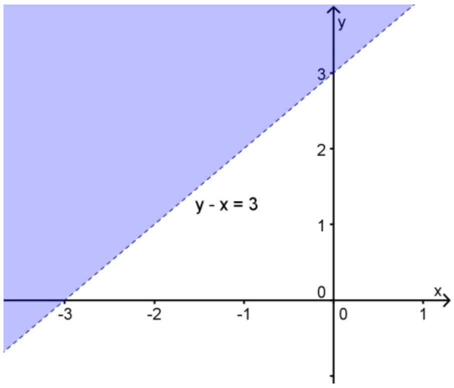
The gradient is therefore equal to 1
$y = x - 2$ (given y –intercept on the graph)
Equation of the shaded region is therefore $y \leq x - 2$

Activity 2

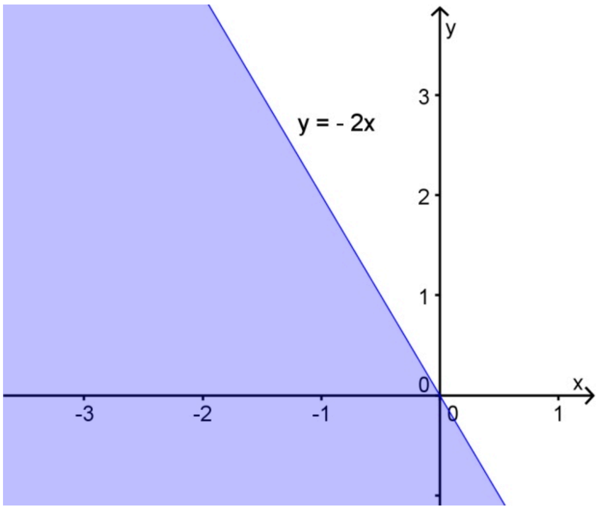
a)



b)

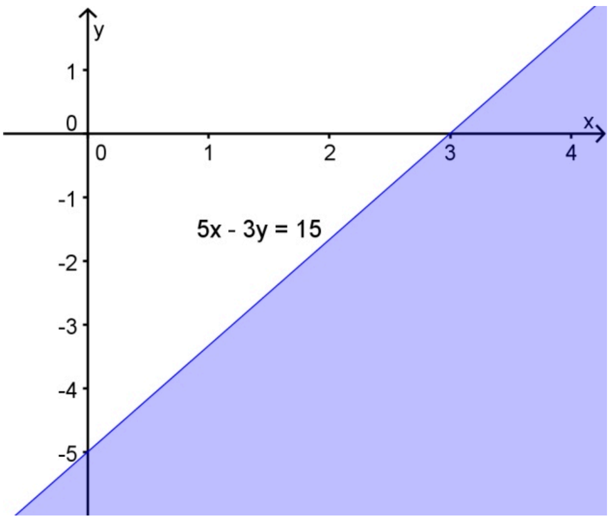


c)



d)

$$5x - 3y \geq 15$$
$$\therefore -3y \geq -5x + 15$$
$$\therefore y \leq \frac{5}{3}x - 5$$

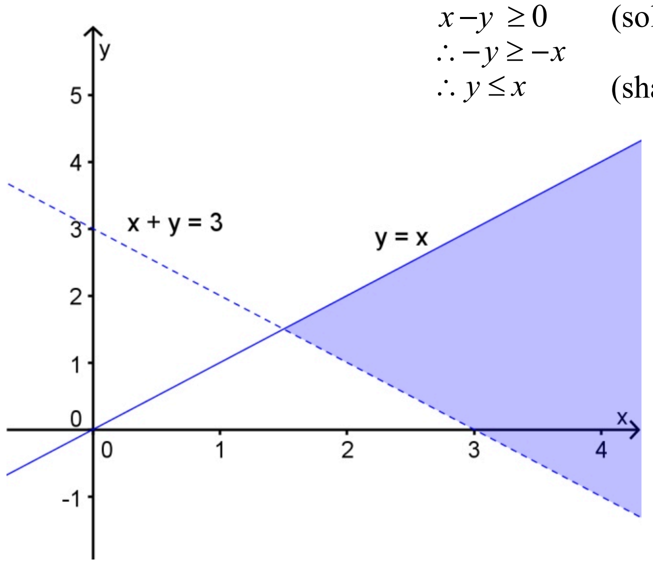


Here we are dividing through by a negative number and so we must also reverse the direction of the inequality side to restore the original balance of the inequality.

Activity 3

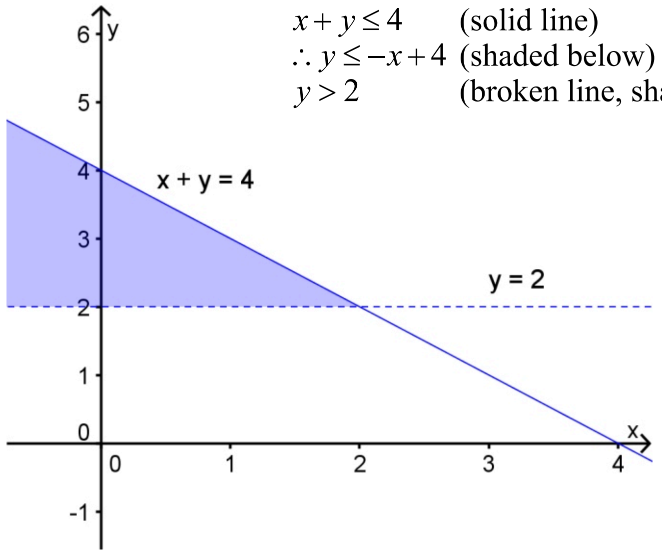
a)

$x + y > 3$ (broken line)
$\therefore y > -x + 3$ (shaded above)
$x - y \geq 0$ (solid line)
$\therefore -y \geq -x$
$\therefore y \leq x$ (shaded below)

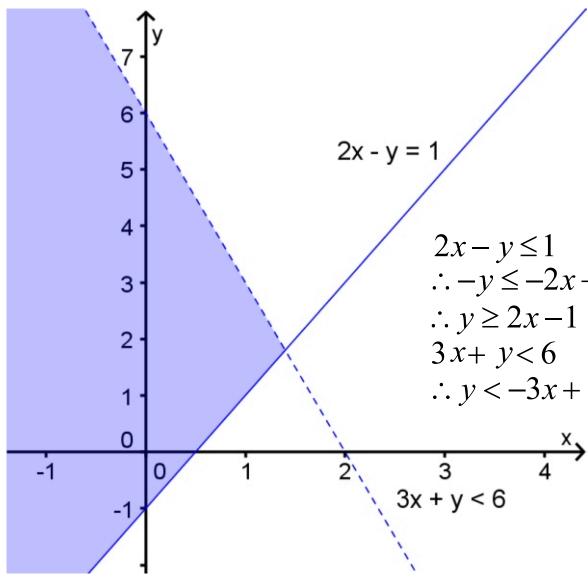


b)

$x + y \leq 4$ (solid line)
$\therefore y \leq -x + 4$ (shaded below)
$y > 2$ (broken line, shaded above)



c)



$2x - y \leq 1$ (solid line)
$\therefore -y \leq -2x + 1$
$\therefore y \geq 2x - 1$ (shaded above)
$3x + y < 6$ (broken line)
$\therefore y < -3x + 6$ (shaded below)

d) $3x + 4y > 12$ (broken line)

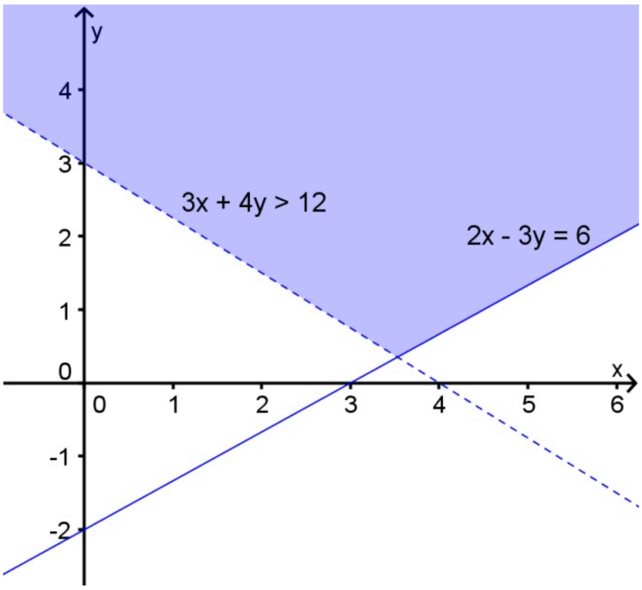
$\therefore 4y > -3x + 12$

$\therefore y > -\frac{3}{4}x + 3$ (shaded above)

$2x - 3y \le 6$ (solid line)

$\therefore -3y \le -2x + 6$

$\therefore y \le \frac{2}{3}x - 2$ (shaded above)



Activity 4

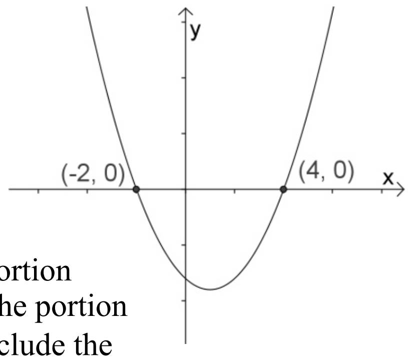
1. $-2x - 8 \ge -x^2$

$\therefore x^2 - 2x - 8 \ge 0$

$\therefore (x-4)(x+2) \ge 0$

$\therefore x \le -2 \quad or \quad x \ge 4$

We are looking for the positive portion of the expression, so we choose the portion that lies above the x –axis. We include the intercepts.

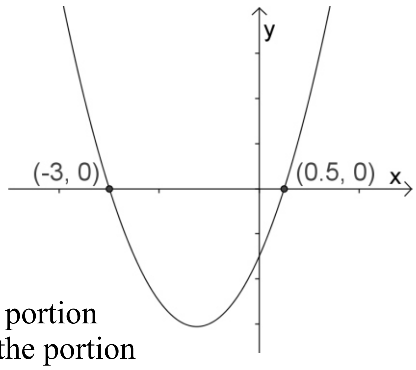


2. $2x^2 - 3 < -5x$

$\therefore 2x^2 + 5x - 3 < 0$

$\therefore (2x-1)(x+3) < 0$

$\therefore -3 < x < \frac{1}{2}$



Looking for the strictly negative portion of the expression, so we choose the portion that lies below the x –axis. We do not include the intercepts.

3. $2\left(x^2+4\right)+10x \le 0$

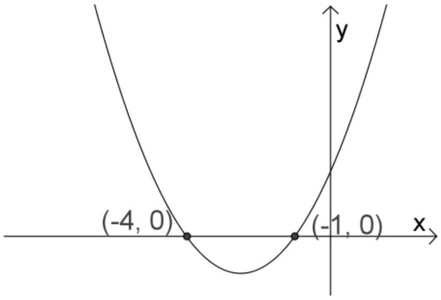
$\therefore 2x^2+8+10x \le 0$

$\therefore 2x^2+10x+8 \le 0$

$\therefore x^2+5x+4 \le 0$

$\therefore (x+4)(x+1) \le 0$

$\therefore -4 \le x \le -1$



4. $7x-x^2 < 12$

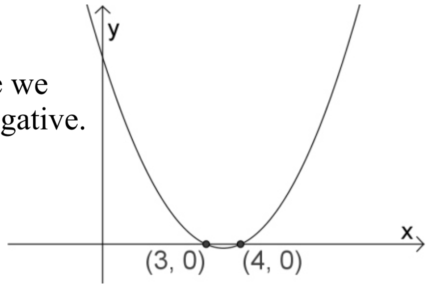
$\therefore -x^2+7x-12 < 0$

$\therefore x^2-7x+12 > 0$

Inequality sign is reversed since we have multiplied through by a negative.

$\therefore (x-4)(x-3) > 0$

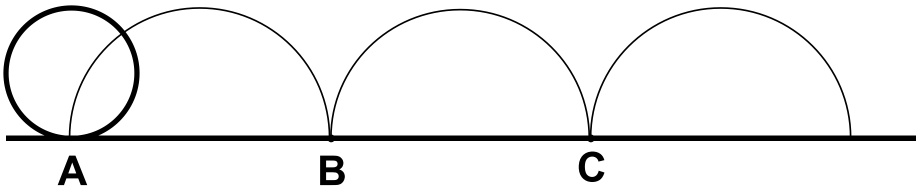
$\therefore x < 3 \quad or \quad x > 4$



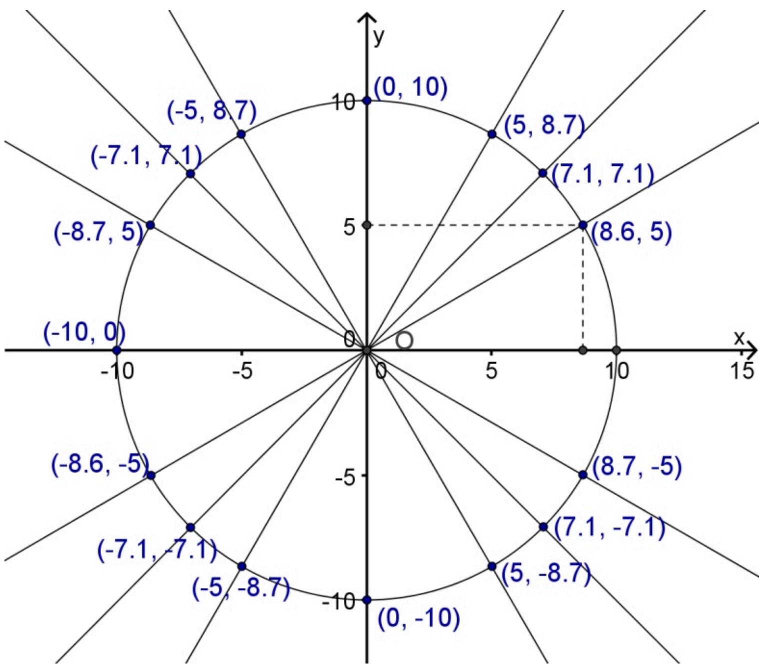
Lesson 3

Activity 1

You probably got something like the figure below. The path of the chalk mark is like arcs that are joined together. The distances from A to B and from B to C are equal. The time taken to move from A to B is called a period. Periodic movement can be seen in many ways.

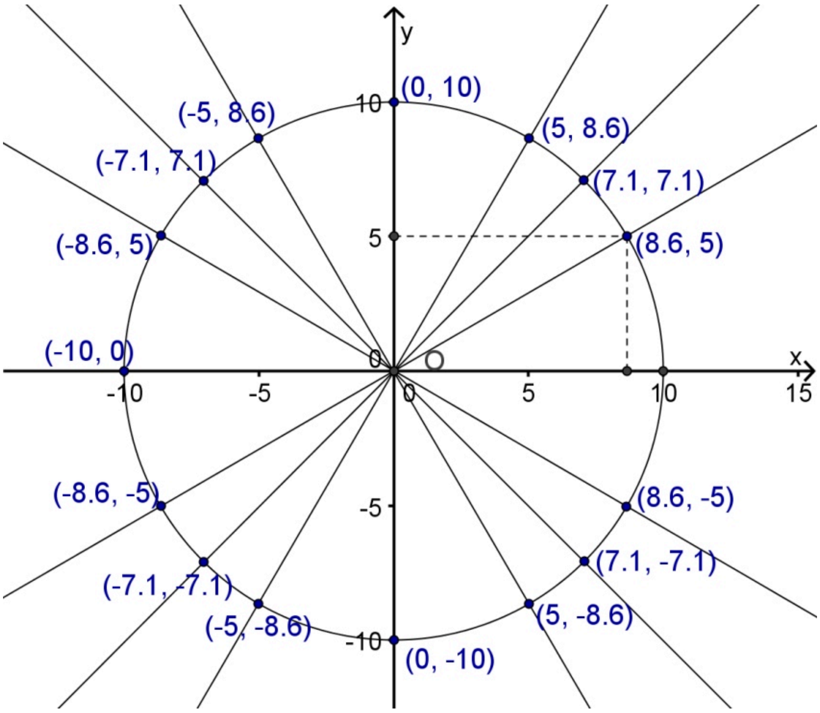


Activity 2



Activity 3

Your drawing will look identical to the drawing in Activity 2 except your radius will be 10 cm instead of 5 cm.



Activity 4

Your readings should be approximately the same as those in the following table. (Note: There are slight inaccuracies due to the crudity of manual measurements.)

Angles	Activity 2 (r = 5)				Activity 3 (r = 10)			
	y	$\frac{y}{r}$	x	$\frac{x}{r}$	y	$\frac{y}{r}$	x	$\frac{x}{r}$
30°	2.5	0.5	4.3	0.86	5	0.5	8.6	0.86
45°	3.5	0.7	3.5	0.7	7.1	0.7	7.1	0.7
60°	4.3	0.86	2.5	0.5	8.6	0.86	5	0.5
90°	5	1	0	0	10	1	0	0
120°	4.3	0.86	-2.5	- 0.5	8.6	0.86	- 5	- 0.5
135°	3.5	0.7	-3.5	- 0.7	7.1	0.7	- 7.1	- 0.7
150°	2.5	0.5	-4.3	-0.86	5	0.5	- 8.6	-0.86
180°	0	0	-5	-1	0	0	- 10	-1
210°	- 2.5	- 0.5	-4.3	- 0.86	-5	- 0.5	- 8.6	- 0.86
225°	- 3.5	- 0.7	-3.5	- 0.7	- 7.1	- 0.7	- 7.1	- 0.7
240°	- 4.3	- 0.86	-2.5	- 0.5	- 8.6	- 0.86	- 5	- 0.5
270°	- 5	-1	0	0	- 10	-1	0	0
300°	- 4.3	- 0.86	2.5	0.5	- 8.6	- 0.86	5	0.5
315°	- 3.5	- 0.7	3.5	0.7	- 7.1	- 0.7	7.1	0.7
330°	- 2.5	- 0.5	4.3	0.86	- 5	- 0.5	8.6	0.86

Activity 5

a) The side adjacent to angle E is DE.
b) The side opposite angle F is DE.
c) The side adjacent to angle G is GI.
d) The side adjacent to angle H is HI.
e) The side opposite angle J is KL.
f) The side adjacent to angle L is KL.
g) The side opposite angle N is OM.
h) The side adjacent to angle M is OM.

Activity 6

a) $\cos A = \cos 30^{\circ}$ $= \frac{AC}{AB}$ $= 0.866$

$\cos D = \cos 30^{\circ}$ $= \frac{DF}{DE}$ $= 0.866$

b) No the measurements of the sides of the triangles are not equal, but the ratios of the measurements are equal.

c) $\sin A = \sin 30^{\circ}$ $= \frac{BC}{AB}$ $= 0.5$

$\sin D = \sin 30^{\circ}$ $= \frac{EF}{DE}$ $= 0.5$

d) No the measurements of the sides of the triangles are not equal, but the ratios of the measurements are equal.

Activity 7

Angles	Activity 2 (r = 5)			Activity 3 (r = 10)		
	y	x	$\frac{y}{x}$	y	x	$\frac{y}{x}$
30°	2.5	4.3	0.58	5	8.6	0.58
45°	3.5	3.5	1	7.1	7.1	1
60°	4.3	2.5	1.72	8.6	5	1.72
90°	5	0	Undefined	10	0	Undefined
120°	4.3	-2.5	- 1.72	8.6	- 5	- 1.72
135°	3.5	-3.5	- 1	7.1	- 7.1	- 1
150°	2.5	-4.3	- 0.58	5	- 8.6	- 0.58
180°	0	-5	0	0	- 10	0
210°	- 2.5	-4.3	0.58	-5	- 8.6	0.58
225°	- 3.5	-3.5	1	- 7.1	- 7.1	1
240°	- 4.3	-2.5	1.72	- 8.6	- 5	1.72
270°	- 5	0	Undefined	- 10	0	Undefined
300°	- 4.3	2.5	- 1.72	- 8.6	5	- 1.72
315°	- 3.5	3.5	- 1	- 7.1	7.1	- 1
330°	- 2.5	4.3	- 0.58	- 5	8.6	- 0.58

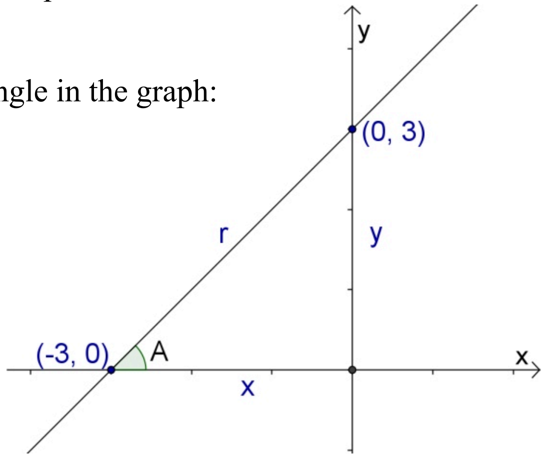
Activity 8

Using a protractor we can measure $\hat{A} = 45^{\circ}$

To find tan A in your calculator push as follows: tan → 45 → =
So $\tan 45^{\circ} = 1$

Using the ratio from the triangle in the graph:

$\tan A = \frac{y}{x} = \frac{3}{3} = 1$



Activity 9

1:1. a) $\sin\alpha = \frac{AB}{AC}$ d) $\sin\beta = \frac{BC}{AC}$

b) $\cos\alpha = \frac{BC}{AC}$ e) $\cos\beta = \frac{AB}{AC}$

c) $\tan\alpha = \frac{AB}{BC}$ f) $\tan\beta = \frac{BC}{AB}$

2. The sum of the angles of a triangle is 180°. One angle is already 90°. Therefore, the other two angles have a sum of 90°. Angles whose sum is 90° are known as complementary angles.

Lesson 4

Activity 1

a) $\sin x = \frac{opp}{hyp}$

$\therefore \sin x = \frac{5}{7}$

$\therefore x = 45{,}58^{\circ}$ shift → sin → $\frac{\square}{\square}$ → 5 → ▼ → 7 → ▶ →) → =

b) $\sin x = \frac{10}{20}$

$\therefore x = 30^{\circ}$ shift → sin → $\frac{\square}{\square}$ → 10 → ▼ → 20 → ▶ →) → =

c) $\sin 60^{\circ} = \frac{d}{5}$

$\therefore 5\sin 60^{\circ} = d$ Multiply both sides by 5

$\therefore d = 4.33$ 5 → x → sin → 60 →) → =

d) $\sin 35^{\circ} = \frac{20}{d}$

$\therefore d\sin 35^{\circ} = 20$ Multiply both sides by d

$\therefore d = \frac{20}{\sin 35^{\circ}}$ Divide both sides by $\sin 35^{\circ}$

$\therefore d = 34{,}87$ ⌸ → 20 → ▼ → sin → 35 →) → =

Activity 2

a) $\cos x = \frac{adj}{hyp}$

$\therefore \cos x = \frac{7}{24}$

$\therefore x = 73^{o}$ shift → cos → ⌸ → 7 → ▼ → 24 → ▶ →) → =

b) $\cos x = \frac{4}{9}$

$\therefore x = 64^{o}$

c) $\cos 40^{\circ} = \frac{adj}{hyp}$

$\therefore \cos 40^{\circ} = \frac{d}{30}$

$\therefore 30\cos 40^{\circ} = d$ multiply both sides by 30

$\therefore d = 22.98$ 30 → x → cos → 40 →) → =

d) $\cos 23^{\circ} = \frac{adj}{hyp}$

$\therefore \cos 23^{\circ} = \frac{100}{d}$

$\therefore d\cos 23^{\circ} = 100$ multiply both sides by d

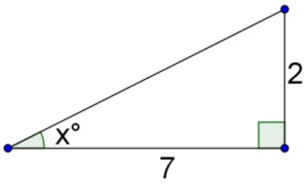
$\therefore d = \frac{100}{\cos 23^{\circ}}$ divide both sides by $\cos 23^{\circ}$

$\therefore d = 108.64$ ⌸ → 100 → ▼ → cos → 23 →) → =

Activity 3

Always draw a right angled triangle with the relevant information on it.

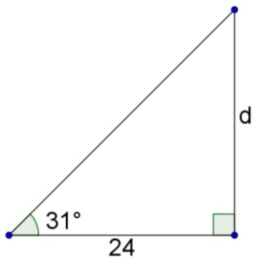
a)



$$\tan x = \frac{2}{7}$$

$\therefore x = 16^{\circ}$ shift → tan → $\frac{\square}{\square}$ → 2 → ▼ → 7 → ▶ →) → =

b)



$$\tan 31^{\circ} = \frac{d}{24}$$

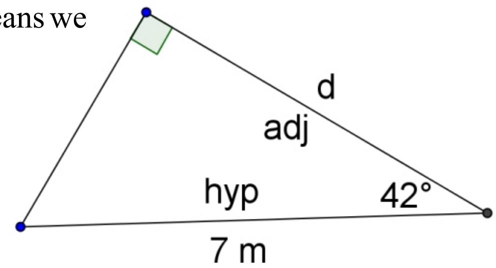
$\therefore 24\tan 31^{\circ} = d$

$\therefore d = 14{,}42$ 24 → x → tan → 31 →) → =

Activity 4

Always start with a sketch of a right angled triangle.

1. The sides involved are the adjacent side and the hypotenuse, this means we must use the cos ratio.



$$\cos 42^{\circ} = \frac{adj}{hyp}$$

$$\therefore \cos 42^{\circ} = \frac{d}{7}$$

$\therefore 7\cos 42^{\circ} = d$

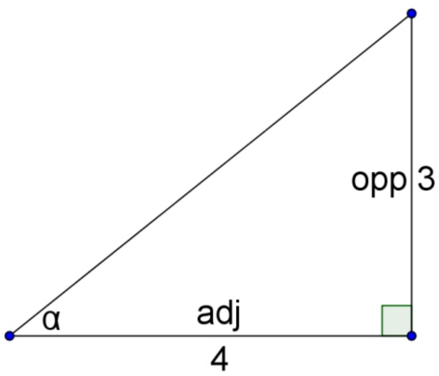
$\therefore d = 5{,}20\ m$

2. The sides involved are the adjacent and opposite sides and the, this means we must use the tan ratio

$$\tan \alpha = \frac{opp}{adj}$$

$$\therefore \tan \alpha = \frac{3}{4}$$

$\therefore \alpha = 36{,}87^{\circ}$



3. The sides involved are the opposite side and the hypotenuse, this means we must use the sin ratio.

$\sin 67^{\circ} = \dfrac{opp}{hyp}$

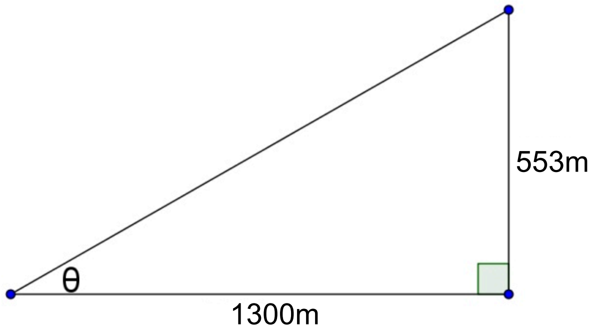
$\therefore \sin 67^{\circ} = \dfrac{d}{30}$

$\therefore 30\sin 67^{\circ} = d$

$\therefore d = 27{,}62\ m$

Activity 5

1.

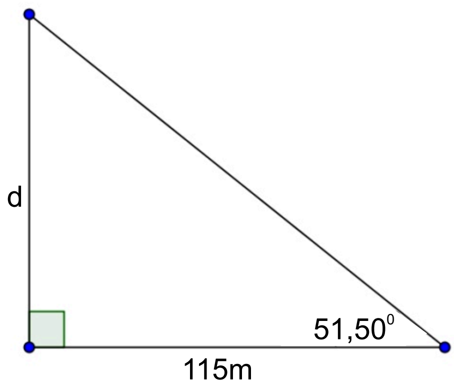


$\tan\theta = \dfrac{opp}{adj}$

$\therefore \tan\theta = \dfrac{553}{1300}$

$\therefore \theta = 23^{\circ}$

2.



$\tan 51{,}50^{\circ} = \dfrac{d}{115}$

$\therefore 115\tan 51{,}50^{\circ} = d$

$\therefore d = 144{,}57\, m$

The pyramid is 144,57 metres tall.

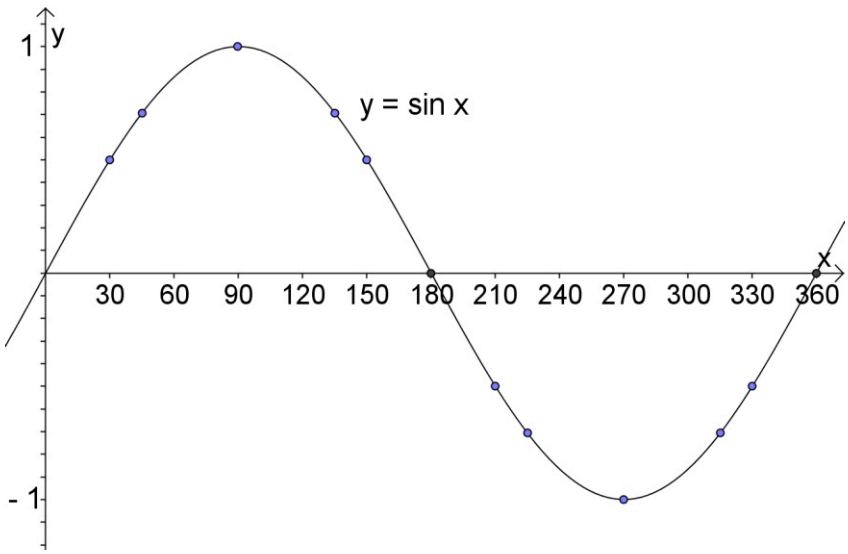
Lesson 5

Activity 1

When we have a curve, things are different. We need more information to find whether the curve has a turning point or not. Some curves, like $y = \sin x$, have more than one turning point. Remember, the only graphs that have straight lines are those where the power of the variable is equal to 1, for example, $y = 2x + 5$. All the other graphs are curved. Here is a better drawing of the graph.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°
$y = \sin x$	0	0,5	0,7	0,87	1	0,87	0,7	0,5	0

x	210°	225°	240°	270°	300°	315°	360°
$y = \sin x$	$-0,5$	$-0,7$	$-0,87$	-1	$-0,87$	$-0,7$	0

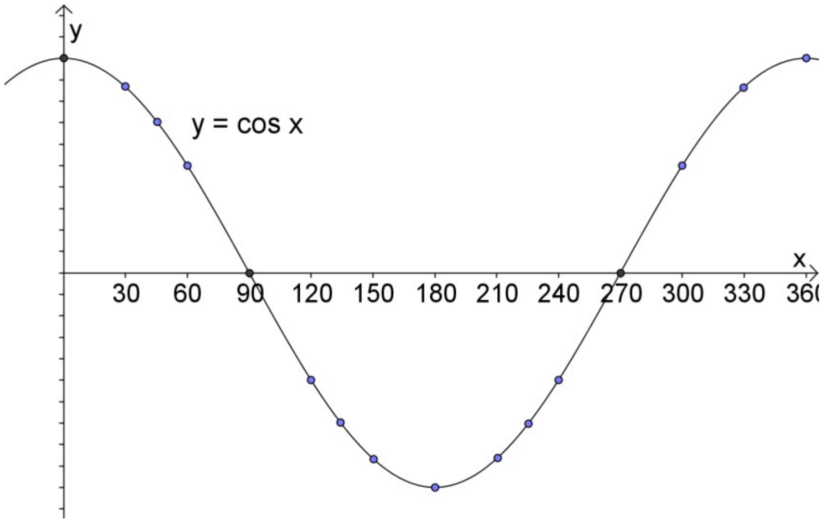


Activity 2

The graph of $y = \cos x$ has the same shape as the one of $y = \sin x$. The only difference is you will see clearly from the table and the graph.

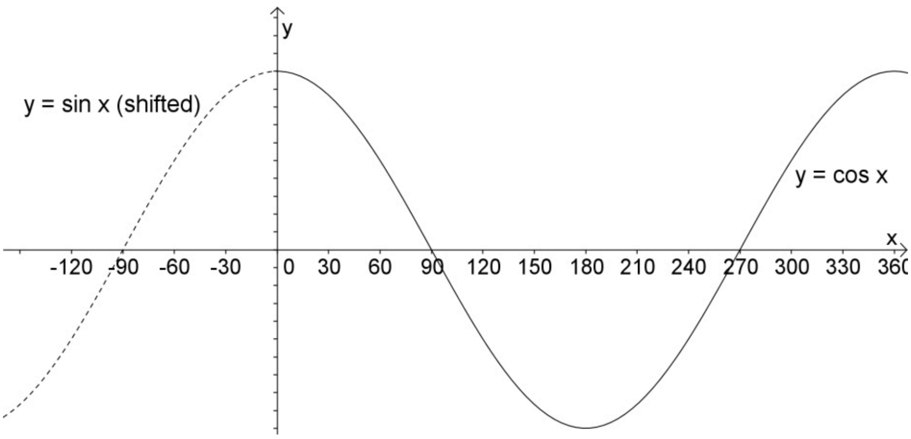
x	0°	30°	45°	60°	90°	120°	135°	150°	180°
$y = \cos x$	1	0,87	0,7	0,5	0	$-0,5$	$-0,7$	$-0,87$	1

x	210°	225°	240°	270°	300°	315°	360°
$y = \cos x$	$-0,87$	$-0,7$	$-0,5$	0	0,5	0,7	1



Activity 3

When the tracing paper of the graph $y = \sin x$ is shifted on top of the graph $y = \cos x$ you must have noted the following: When the 90° mark on the tracing paper ($y = \sin x$) coincides with the 0° mark of the graph of Activity 2 ($y = \cos x$), the two graphs are the same, they coincide on all points. The following drawing illustrates this.

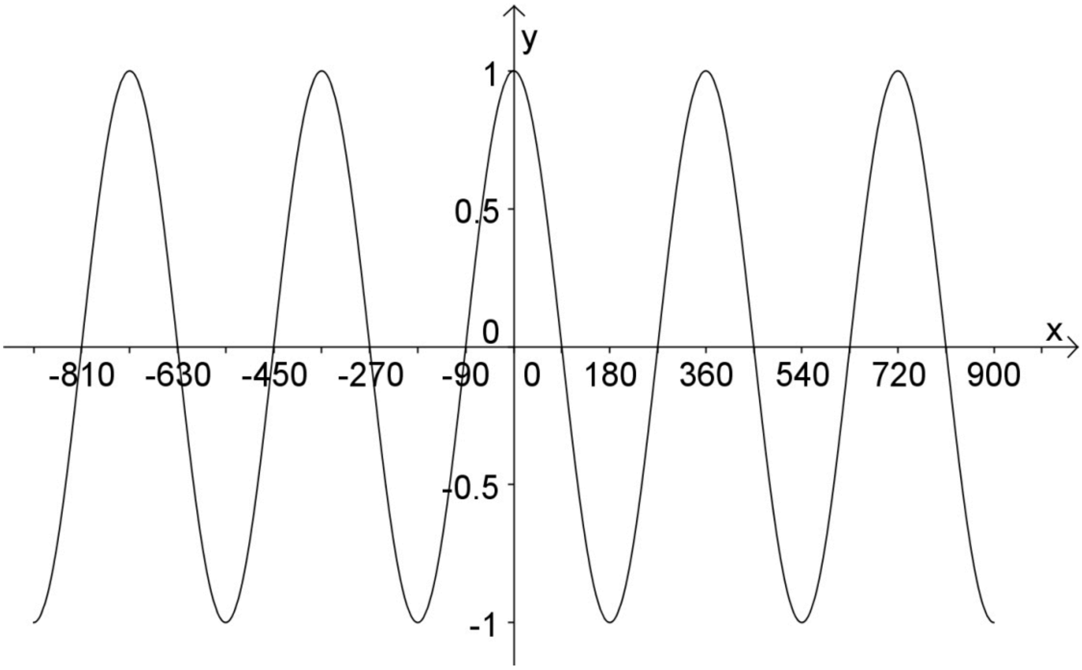


Activity 4

$\sin 30^\circ = 0{,}5$	$\cos 30^\circ = 0{,}87$	$\tan 30^\circ = 0{,}58$
$\sin(-30^\circ) = -0{,}5$	$\cos(-30^\circ) = 0{,}87$	$\tan(-30^\circ) = -0{,}58$
$\sin 60^\circ = 0{,}87$	$\cos 60^\circ = 0{,}5$	$\tan 60^\circ = 1{,}73$
$\sin(-60^\circ) = -0{,}87$	$\cos(-60^\circ) = 0{,}5$	$\tan(-60^\circ) = -1{,}73$
$\sin 150^\circ = 0{,}5$	$\cos 150^\circ = -0{,}87$	$\tan 150^\circ = -0{,}58$
$\sin(-150^\circ) = -0{,}5$	$\cos(-150^\circ) = -0{,}87$	$\tan(-150^\circ) = 0{,}58$
$\sin 225^\circ = -0{,}71$	$\cos 225^\circ = -0{,}71$	$\tan 225^\circ = 1$
$\sin(-225^\circ) = 0{,}71$	$\cos(-225^\circ) = -0{,}71$	$\tan(-225^\circ) = -1$

Activity 5

Your graph will look like this:



Activity 6

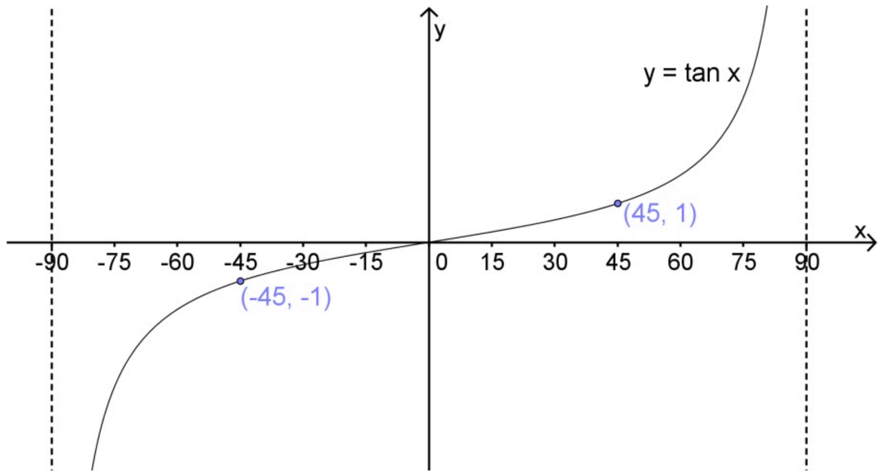
x^o	-90^o	-75^o	-60^o	-45^o	-30^o	-15^o	0^o	15^o	30^o	45^o	60^o	75^o	90^o
$y = \tan x$	?	–3,7	–1,7	–1	–0,6	–0,3	0	0,3	0,6	1	1,7	3,7	?

Activity 7

x^o	75^o	80^o	85^o	86^o	87^o	88^o	89^o	$89{,}5^o$	$89{,}8^o$	$89{,}9^o$	90^o
$y = \tan x$	3,7	5,7	11,4	14,3	19,1	28,6	57,3	114,6	286,5	573	?

Activity 8

The graph of $y = \tan x$, where x is between $-90°$ and $90°$ is given below.



When a function is not defined for a given value, this is shown in the graph by a dotted line, in this case, parallel to the y –axis. This line is called an asymptote. Since $\tan 90^{o}$ and $\tan\left(-90^{o}\right)$ do not exist, we draw dotted lines at both values.

Feedback to Self-Check Exercises

Lesson 1

1. Consider the function $y = \dfrac{3}{x-1} - 2$

a) x - int: $2 = \dfrac{3}{x-1}$ $\qquad$ y - int: $y = \dfrac{3}{0-1} - 2 = -5$

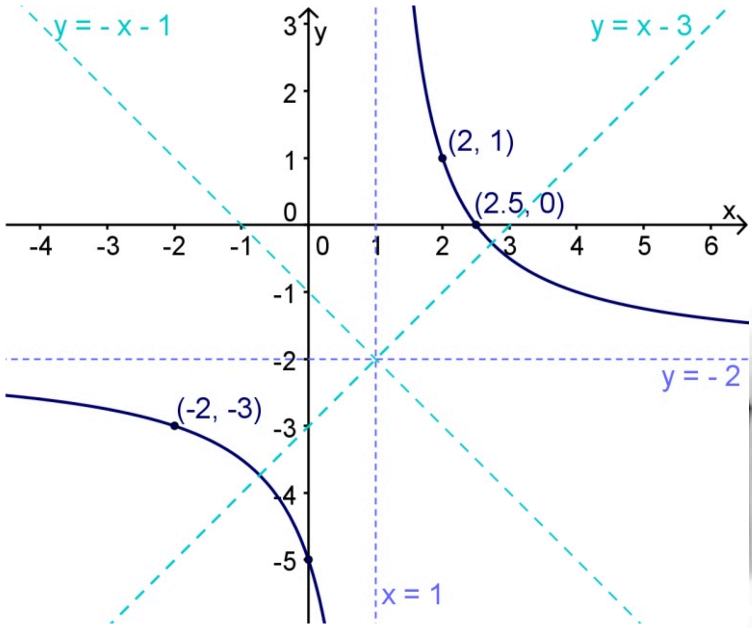
$\therefore 2x - 2 = 3$

$\therefore 2x = 5$

$\therefore x = \dfrac{5}{2}$

b) $x = 1,\ y = -2$

c)



d) $x \in \mathbb{R}, x \neq 1$

2. Asymptotes are at $x = 1$ and $y = 2$

$\therefore y = \dfrac{a}{x-1} + 2$

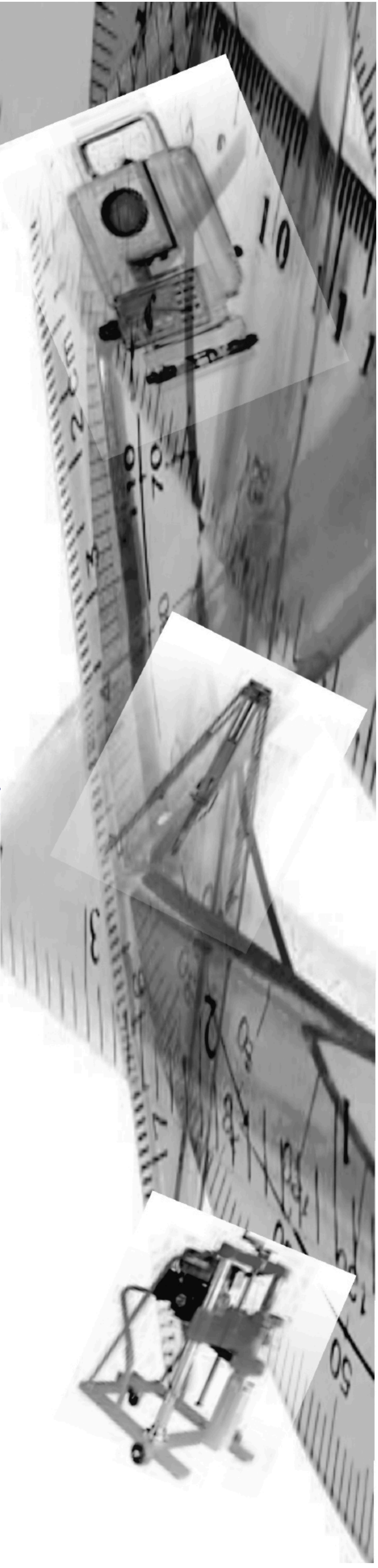
Substitute the point $(-3;1)$

$\therefore 1 = \dfrac{a}{-3-1} + 2$

$\therefore -1 = -\dfrac{a}{4}$

$\therefore a = 4$

$\therefore y = \dfrac{4}{x-1} + 2$



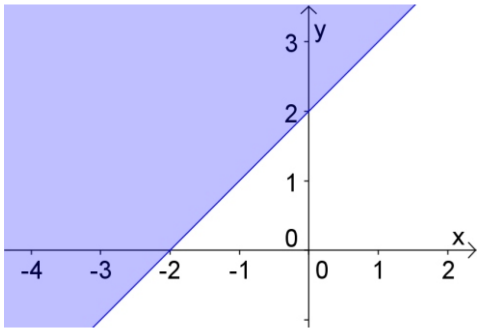
Lesson 2

1. a) $y = x + 2$

y –intercept put $x = 0$ $\therefore y = 2$ $\therefore (0; 2)$

x –intercept put $y = 0$ $\therefore x + 2 = 0$ $\therefore x = -2$

$\therefore (-2; 0)$

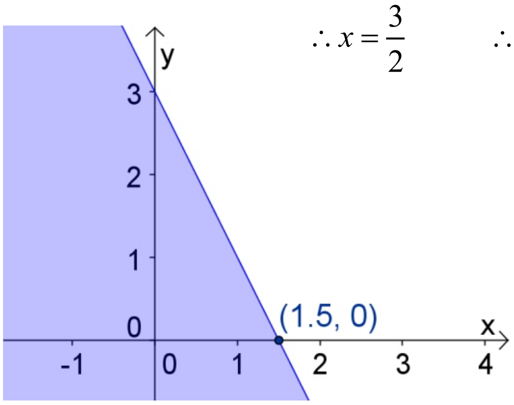


b) $y = -2x + 3$

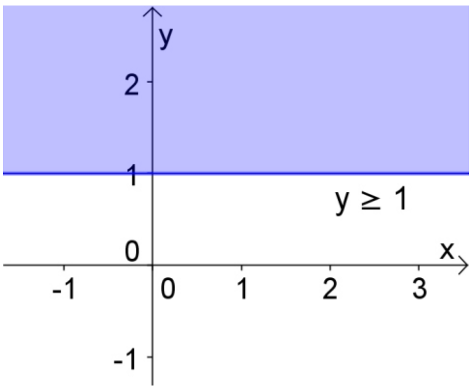
y –intercept put $x = 0$ $\therefore y = 3$ $\therefore (0; 3)$

x –intercept put $y = 0$ $\therefore -2x + 3 = 0$

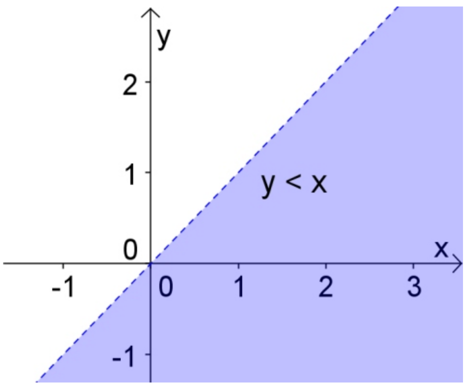
$\therefore x = \frac{3}{2}$ $\therefore \left(\frac{3}{2}; 0\right)$



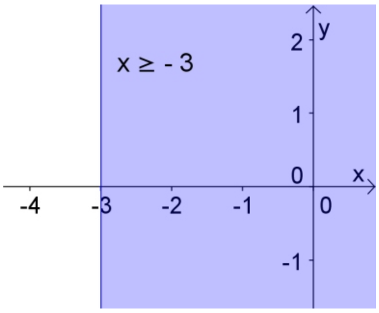
2. a)



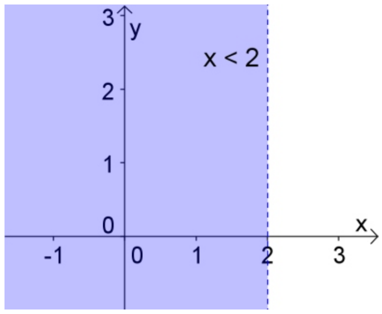
b)



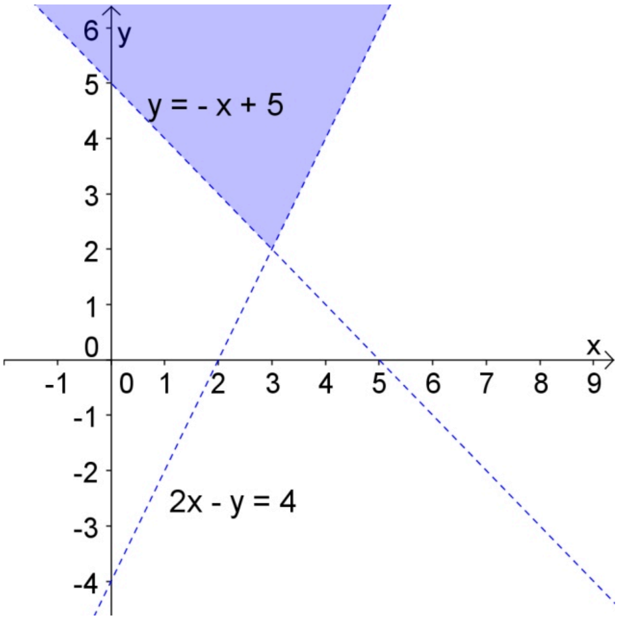
c)



d)



3.

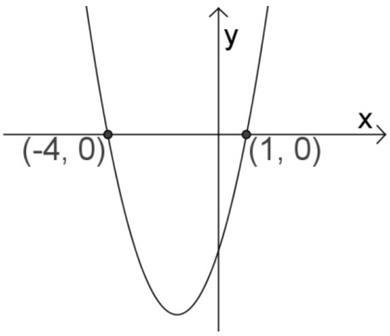


4. $-x^2 - 3x + 4 \geq 0$

$\therefore x^2 + 3x - 4 \leq 0$

$\therefore (x+4)(x-3) \leq 0$

$\therefore -4 \leq x \leq 1$



Lesson 3

1. A window wiper of a car is an example of an object that has periodic movement. The movement is periodic because the wiper will go from left to right and back at given intervals. This one is very interesting because the periods can be adjusted. When the wiper is slow, the period is greater. The faster the wiper moves, the smaller the period it takes to go from left to right and back. (Most car wipers start from the left.)

   There are many examples of wheel movements. These are all periodic movements.

2. $\sin A = \dfrac{opposite}{hypotenuse} = \dfrac{y}{r}$

$\cos A = \dfrac{adjacent}{hypotenuse} = \dfrac{x}{r}$

$\tan\alpha = \dfrac{opposite}{adjacent} = \dfrac{y}{x}$

3. $\sin\alpha = \dfrac{2.6}{5} = 0.52$

$\cos\alpha = \dfrac{4.3}{5} = 0.86$

$\tan\alpha = \dfrac{2.6}{4.3} = 0.6$

$\sin\beta = \dfrac{4.5}{5} = 0.9$

$\cos\beta = \dfrac{-2.2}{5} = -0.44$

$\tan\beta = \dfrac{4.5}{-2.2} = -2.045$

$\sin\theta = \dfrac{-3.8}{5} = -.076$

$\cos\theta = \dfrac{-3.2}{5} = -0.64$

$\tan\theta = \dfrac{-3.8}{-3.2} = 1.2$

Lesson 4

1. a) $\cos 40^{\circ} = \dfrac{8}{AC}$

$\therefore AC\cos 40^{\circ} = 8$

$\therefore AC = \dfrac{8}{\cos 40^{\circ}}$

$\therefore AC = 10{,}44$

b) $\tan 30^{\circ} = \dfrac{3}{DE}$

$\therefore DE\tan 30^{\circ} = 3$

$\therefore DE = \dfrac{3}{\tan 30^{\circ}}$

$\therefore DE = 5{,}20$

c) $\sin 30^\circ = \frac{3}{DF}$

$\therefore DF \sin 30^\circ = 3$

$\therefore DF = \frac{3}{\sin 30^\circ}$

$\therefore DF = 6$

d) $\sin 80^\circ = \frac{GH}{10}$

$\therefore GH = 10\sin 80^\circ$

$\therefore GH = 9{,}85$

e) $\cos 80^\circ = \frac{HI}{10}$

$\therefore HI = 10\cos 80^\circ$

$\therefore HI = 1{,}74$

f) $\tan 45^\circ = \frac{JK}{6}$

$\therefore JK = 6\tan 45^\circ$

$\therefore JK = 6$

g) $\cos 45^\circ = \frac{6}{JL}$

$\therefore JL \cos 45^\circ = 6$

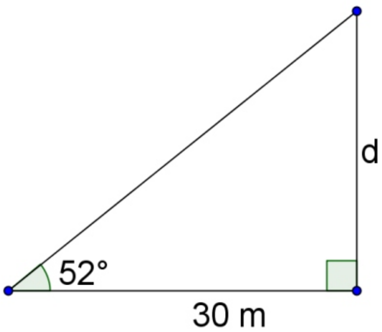
$\therefore JL = \frac{6}{\cos 45^\circ}$

$\therefore JL = 8{,}49$

2. $\tan 52^\circ = \frac{d}{30}$

$\therefore d = 30\tan 52^\circ$

$\therefore d = 38{,}40$



Don't forget to add the height of the person which is 1,5 m to the height of BC (38,40 m) to get the height of the church spire.

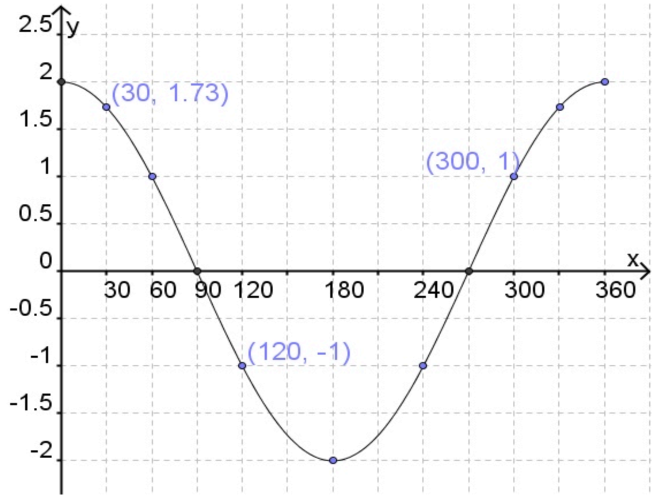
$\text{height} = 38{,}40 + 1{,}5 = 39{,}90$

To the nearest 10 cm, height is 3990 cm

Lesson 5

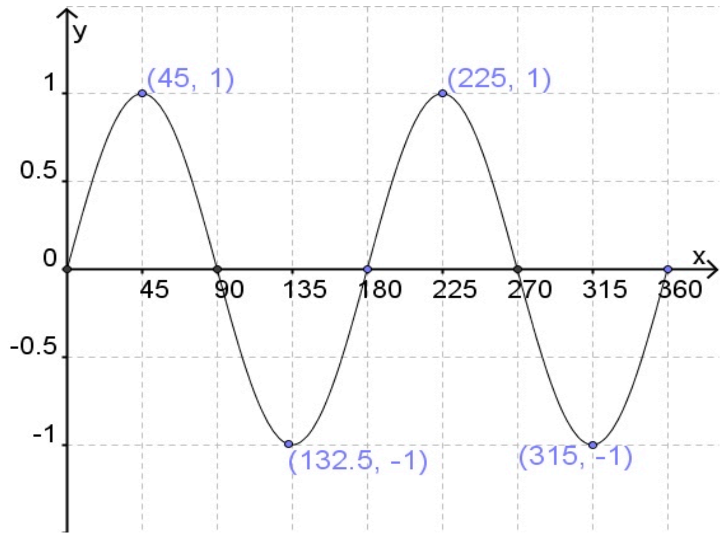
1. a)

x°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos x$	0,87	0,5	0	–0,5	–0,87	–1	–0,87	–0,5	0	0,5	0,87	1
$2\cos x$	1,73	1	0	–1	–1,73	–2	–1,73	–1	0	1	1,73	2

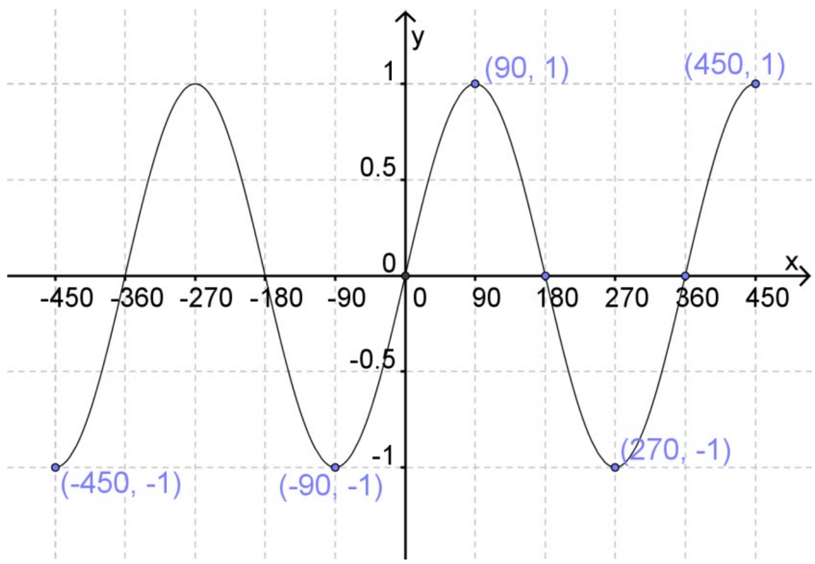


b)

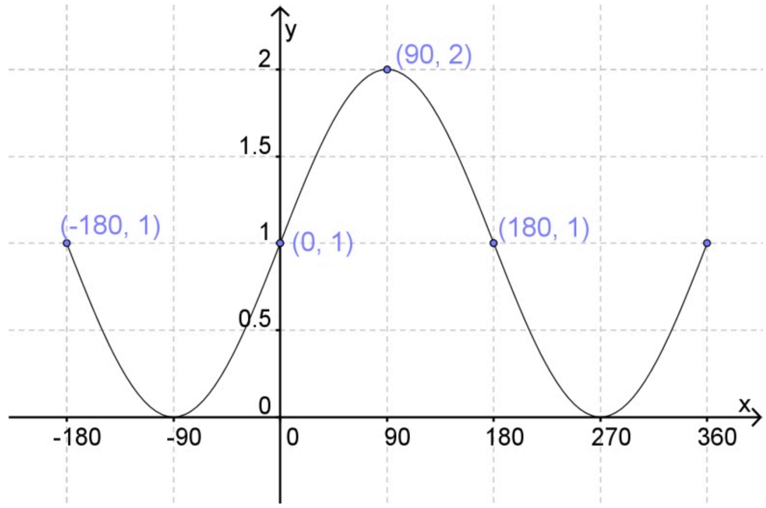
x^o	45^o	90^o	135^o	180^o	225^o	270^o	315^o	360^o
$2x$	90^o	180^o	270^o	360^o	450^o	540^o	630^o	720^o
$\sin 2x$	1	0	-1	1	0	-1	0	1



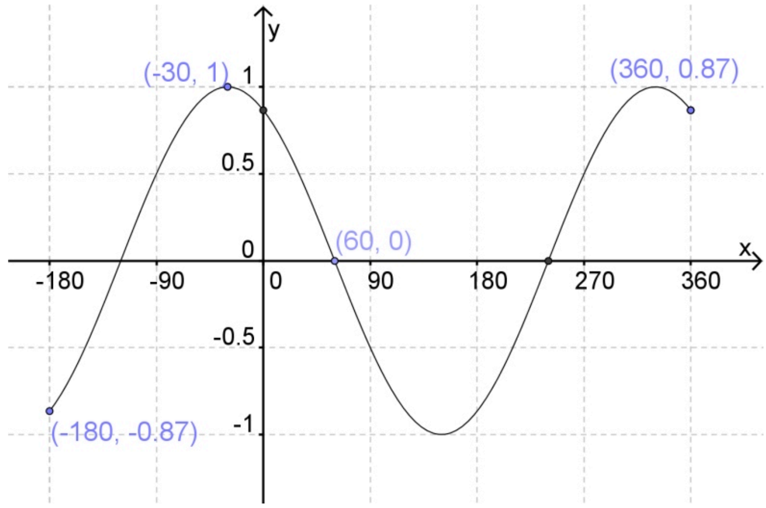
2.



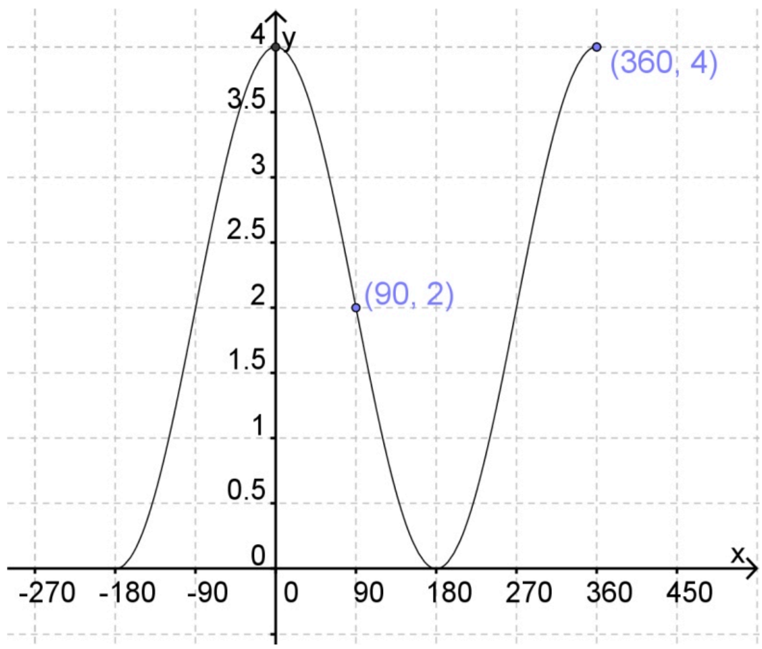
3. a)


y
x
2
1.5
1
0.5
0
-180
-90
0
90
180
270
360
(90, 2)
(-180, 1)
(0, 1)
(180, 1)

b)


y
x
1
0.5
0
-0.5
-1
-180
-90
0
90
180
270
360
(-30, 1)
(360, 0.87)
(60, 0)
(-180, -0.87)

c)


y
x
4
3.5
3
2.5
2
1.5
1
0.5
0
-270
-180
-90
0
90
180
270
360
450
(360, 4)
(90, 2)

4. $\tan 450^{\circ} = \tan 90$

A full revolution has taken place. The angle has moved past 360° and moved onto 90°.

$\tan 90^{\circ}$ is undefined